ALPHA-DISTANCE – A GENERALIZATION OF CHINESE CHECKER DISTANCE AND TAXICAB DISTANCE

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The Euclidean distance measures the shortest distance between two points. The taxicab distance [2] measures the distance between two points when only moves along axis-directions are permitted. If, in addition, the diagonal moves are also permitted, the distance between A and B is given by the Chinese checker distance [1]. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in $\mathbb{R}^2$. Denote

$$\Delta_{AB} = \max\{|x_2 - x_1|, |y_2 - y_1|\} \quad \text{and} \quad \delta_{AB} = \min\{|x_2 - x_1|, |y_2 - y_1|\}.$$

The Euclidean distance between A and B is

$$d(A, B) = \sqrt{\Delta_{AB}^2 + \delta_{AB}^2}\ .$$

The taxicab distance between A and B is

$$d_T(A, B) = \Delta_{AB} + \delta_{AB}.$$

The Chinese checker distance between A and B is

$$d_C(A, B) = \Delta_{AB} + (\sqrt{2} - 1)\delta_{AB}.$$

In this paper we introduce a family of distances which include both Chinese checker distance and taxicab distance as special cases. For each $\alpha \in [0, \pi/4]$, we define the α-distance between A and B by

$$d_\alpha(A, B) = \Delta_{AB} + (\sec\alpha - \tan\alpha)\delta_{AB}.$$

Then, $d_0(A, B) = d_T(A, B)$ and $d_{\pi/4}(A, B) = d_C(A, B)$. Observe that if $\delta_{AB} > 0$, then

$$d_C(A, B) < d_\alpha(A, B) < d_T(A, B) \quad \text{for all} \quad \alpha \in (0, \pi/4).$$

When $\delta_{AB} = 0$, A and B lie on a horizontal or vertical line, and it follows that $d_C(A, B) = d_\alpha(A, B) = d_T(A, B) = d(A, B)$ for all $\alpha \in [0, \pi/4]$.

Clearly, $d_\alpha(A, B) = 0$ if and only if $A = B$, and $d_\alpha(A, B) = d_\alpha(B, A)$ for all $A, B \in \mathbb{R}^2$. The main result of this paper will be that

$$d_\alpha(A, B) \le d_\alpha(A, C) + d_\alpha(C, B),$$

for all $A, B, C \in \mathbb{R}^2$, and $\alpha \in [0, \pi/4]$.

For two points $A(x_1, y_1), B(x_2, y_2) \in \mathbb{R}^2$, we denote by R_{AB} the rectangular region bounded by the lines $x = x_1$, $y = y_1$, $x = x_2$, and $y = y_2$. The following propositions follow directly from the definition of the α-distance.

Proposition 1. Let A, B, C, and D be four points in $\mathbb{R}^2$. If R_{AB} and R_{CD} are congruent, then

$$d_\alpha(A, B) = d_\alpha(C, D) \text{ for all } \alpha \in [0, \pi/4].$$

As a consequence of Proposition 1, the α-distance between two points is an invariant under translations and reflections about the x-axis, the y-axis, or both.

Proposition 2. Let A and B be two points in $\mathbb{R}^2$. Then

$$d_\alpha(A, B) \geq d_\alpha(C, D) \text{ for all } C, D \in R_{AB} \text{ and } \alpha \in [0, \pi/4].$$

The following lemma is useful in establishing the main result.

Lemma 3. If $\alpha \in [0, \pi/4]$, then

$$\Delta_{AB} + (\sec\alpha - \tan\alpha)\delta_{AB} \geq \delta_{AB} + (\sec\alpha - \tan\alpha)\Delta_{AB}$$

for all $A, B \in \mathbb{R}^2$.

Proof. If $\Delta_{AB} = \delta_{AB}$, the result holds. Assume that $\Delta_{AB} > \delta_{AB}$. Since $\alpha \in [0, \pi/4]$, it follows that $\cos\alpha > 0$ and $\sin\alpha + \cos\alpha \geq 1$. Therefore, $\tan\alpha + 1 \geq \sec\alpha$, i.e., $1 \geq \sec\alpha - \tan\alpha$. Multiplying both sides by $\Delta_{AB} - \delta_{AB}$, we have

$$\begin{gathered} \Delta_{AB} - \delta_{AB} \geq (\Delta_{AB} - \delta_{AB})(\sec\alpha - \tan\alpha), \quad \text{i.e.,} \\ \Delta_{AB} + (\sec\alpha - \tan\alpha)\delta_{AB} \geq \delta_{AB} + (\sec\alpha - \tan\alpha)\Delta_{AB}. \end{gathered}$$

Theorem 4. Let A and B be two points in $\mathbb{R}^2$ and $\alpha \in [0, \pi/4]$. Then,

$$d_\alpha(A, B) \leq d_\alpha(A, C) + d_\alpha(C, B),$$

for all $C \in \mathbb{R}^2$.

Proof. It is obvious that the result holds when $\delta_{AB} = 0$. Assume $\delta_{AB} > 0$. By Proposition 1, without loss of generality, assume that A lies at the origin, and $B(x_2, y_2)$, with $x_2 > y_2 > 0$, lies in the first quadrant. Identify points $D(x_2 - y_2, 0)$,

$E(x_2 - y_2 \tan\alpha, 0)$, and $F(x_2, 0)$. Let $R_1 = \Delta ABD$, $R_2 = \Delta DBE$, $R_3 = \Delta EBF$, and

$$R_4 = \{(x, y) \mid x \geq 0 \text{ and } y < 0, \text{ or } x > x_2 \text{ and } y \geq 0\}$$

(see Figure 1).

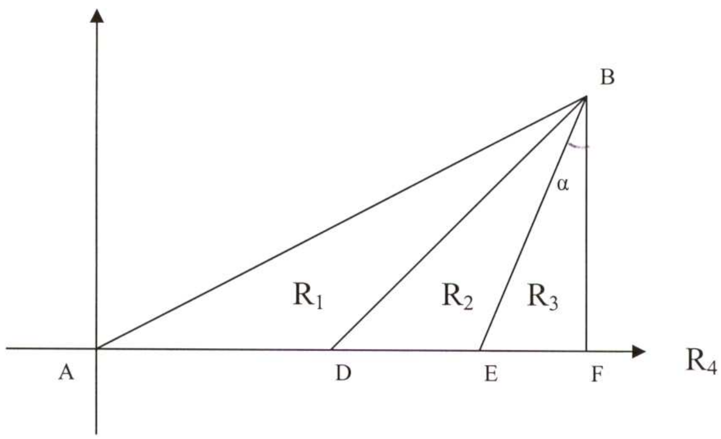

Figure 1.

By Proposition 1, it suffices to prove the result for $C \in R_1 \cup R_2 \cup R_3 \cup R_4$.

Case 1. Assume that $C(x, y) \in R_1$.
In this case, it follows that $x \geq y$ and $x_2 - x \geq y_2 - y$. Thus,

$$d_\alpha(A, C) = x + (\sec\alpha - \tan\alpha)y \quad \text{and} \quad d_\alpha(C, B) = x_2 - x + (\sec\alpha - \tan\alpha)(y_2 - y).$$

Therefore,

$$d_\alpha(A, C) + d_\alpha(C, B) = x_2 + (\sec\alpha - \tan\alpha)y_2 = d_\alpha(A, B).$$

Case 2. Assume $C(x, y) \in R_2$.
Draw a line segment parallel to the x-axis through C to intersect line segment $\overline{BE}$ at G. Draw a line segment parallel to $\overline{BE}$ through C to intersect the x-axis at point H (see Figure 2).

Since the quadrilateral $CGEH$ is a parallelogram, it follows that $|\overline{CG}| = |\overline{HE}|$ and $|\overline{HC}| = |\overline{EG}|$. Note that $d_\alpha(A,C) = |\overline{AH}| + |\overline{HC}|$ and

$$\begin{aligned} d_\alpha(A,B) &= |\overline{AE}| + |\overline{EB}| \\ &= |\overline{AH}| + |\overline{HE}| + |\overline{EG}| + |\overline{GB}| \\ &= |\overline{AH}| + |\overline{HC}| + |\overline{CG}| + |\overline{GB}| \\ &= d_\alpha(A,C) + |\overline{CG}| + |\overline{GB}|. \end{aligned}$$

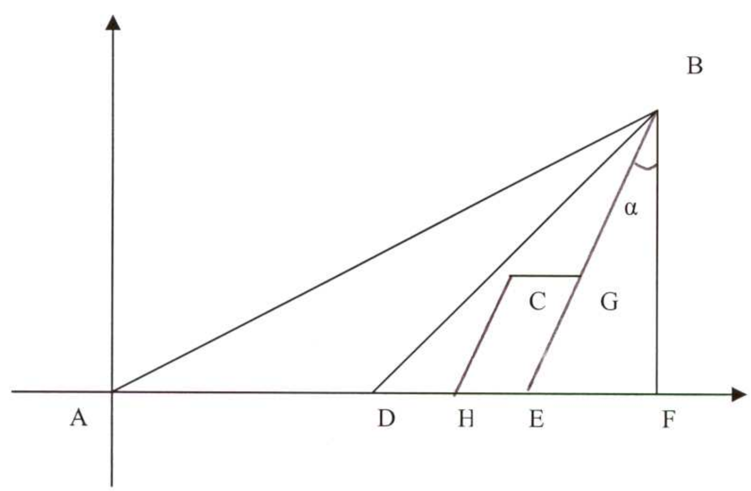

Figure 2.

By Lemma 3,

$$|\overline{CG}| + |\overline{GB}| \le d_\alpha(C,B).$$

Therefore,

$$d_\alpha(A,B) \le d_\alpha(A,C) + d_\alpha(C,B).$$

Case 3. Assume that $C(x,y) \in R_3$.

Let G and H be the points obtained as in Case 2. As in Case 2, it follows that $|\overline{GC}| = |\overline{EH}|$ and $|\overline{HC}| = |\overline{EG}|$. Observe that,

$$\begin{aligned} d_\alpha(A,B) &= |\overline{AE}| + |\overline{EB}| \\ &= |\overline{AE}| + |\overline{EG}| + |\overline{GB}|, \\ d_\alpha(A,C) &= |\overline{AH}| + |\overline{HC}| \\ &= |\overline{AE}| + |\overline{EH}| + |\overline{HC}| \\ &= |\overline{AE}| + |\overline{EG}| + |\overline{GC}|, \end{aligned}$$

and

$$|\overline{CB}| \leq d_\alpha(C,B).$$

Therefore,

$$\begin{aligned} d_\alpha(A,B) &= |\overline{AE}| + |\overline{EG}| + |\overline{GB}| \\ &\leq |\overline{AE}| + |\overline{EG}| + |\overline{GC}| + |\overline{CB}| \\ &= d_\alpha(A,C) + d_\alpha(C,B). \end{aligned}$$

<u>Case 4</u>. Assume that $C(x,y) \in R_4$.

Let $h = \min\{\max\{0,y\}, y_2\}$ and $k = \min\{x, x_2\}$. Denote by D the point (h,k). Then D lies on the line segment $\overline{AF}$ or $\overline{BF}$ (see Figure 1). By Proposition 2,

$$d_\alpha(A,D) \leq d_\alpha(A,C) \quad \text{and} \quad d_\alpha(D,B) \leq d_\alpha(C,B).$$

Based on the results from Case 2 and Case 3,

$$d_\alpha(A,B) \leq d_\alpha(A,D) + d_\alpha(D,B) \leq d_\alpha(A,C) + d_\alpha(C,B).$$

We know that geometric figures such as lines, circles, parabolas, ellipses, and hyperbolas can be defined based on a distance. It is a good undergraduate/graduate research topic to characterize these figures under the α-distance.

References

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