INFINITELY MANY COMPOSITE NSW NUMBERS: AN INDUCTIVE PROOF

James A. Sellers

1. Motivation. The NSW numbers were introduced approximately 20 years ago [3] in connection with the order of certain simple groups. These are the numbers f_n which satisfy the recurrence

$$f_{n+1} = 6f_n - f_{n-1} \tag{1}$$

with initial conditions $f_1 = 1$ and $f_2 = 7$.

In recent years, these numbers have been studied from a variety of perspectives [1, 2]. Moreover, the author, in collaboration with Hugh Williams, has proven that there are infinitely many composite NSW numbers [4] as requested in [1]. The goal of this note is to provide a purely inductive proof of the main theorem in [4]. We restate it here.

Theorem 1.1. For all $m \geq 1$ and all $n \geq 0$, $f_m | f_{(2m-1)n+m}$.

2. The Necessary Tools. To prove Theorem 1.1, we need to develop a few key tools.

Proposition 2.1. For all integers $a, b \geq 0$, and for all $1 \leq j \leq a+b-2$, we have

$$f_{a+b} = s_{j+1} f_{a+b-j} - s_j f_{a+b-j-1} \tag{2}$$

where

$$s_j = \sum_{i=1}^{j} (-1)^{i+j} f_i.$$

Proof. We prove this proposition using induction on j. First, when $j = 1$, the right hand side of (2) is $(f_2 - f_1) f_{a+b-1} - f_1 f_{a+b-2}$ or $6 f_{a+b-1} - f_{a+b-2}$, which equals f_{a+b} thanks to (1).

Next, we assume

$$f_{a+b} = s_{j+1} f_{a+b-j} - s_j f_{a+b-j-1}$$

for $j < a + b - 2$. Thus, since $f_{a+b-j} = 6f_{a+b-j-1} - f_{a+b-j-2}$, we have

$$\begin{aligned} f_{a+b} &= s_{j+1}(6f_{a+b-j-1} - f_{a+b-j-2}) - s_j f_{a+b-j-1} \\ &= (6s_{j+1} - s_j) f_{a+b-j-1} - s_{j+1} f_{a+b-j-2}. \end{aligned}$$

Then we note that by (1)

$$\begin{aligned} 6s_{j+1} - s_j &= 6\sum_{i=1}^{j+1}(-1)^{i+j+1} f_i - \sum_{i=1}^{j}(-1)^{i+j} f_i \\ &= 6(-1)^{j+2} f_1 + 6\sum_{i=2}^{j+1}(-1)^{i+j+1} f_i - \sum_{i=1}^{j}(-1)^{i+j} f_i \\ &= 6(-1)^{j+2} f_1 + 6\sum_{i=1}^{j}(-1)^{i+j+2} f_{i+1} - \sum_{i=1}^{j}(-1)^{i+j} f_i \\ &= (f_2 - f_1)(-1)^{j+2} + \sum_{i=1}^{j}(-1)^{i+j+2}(6f_{i+1} - f_i) \\ &= (f_2 - f_1)(-1)^{j+2} + \sum_{i=1}^{j}(-1)^{i+j+2} f_{i+2} \\ &= (f_2 - f_1)(-1)^{j+2} + \sum_{i=3}^{j+2}(-1)^{i+j} f_i \\ &= \sum_{i=1}^{j+2}(-1)^{i+j} f_i \\ &= s_{j+2}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} f_{a+b} &= (6s_{j+1} - s_j)f_{a+b-j-1} - s_{j+1}f_{a+b-j-2} \\ &= s_{j+2}f_{a+b-j-1} - s_{j+1}f_{a+b-j-2}, \end{aligned}$$

which completes the proof of Proposition 2.1.

Proposition 2.2. For all $m \geq 1$ and for all $1 \leq c \leq m-1$, $f_m | f_{m+c} + f_{m-c}$.

Proof. For $c = 1$, we know from (1) that $6f_m = f_{m+1} + f_{m-1}$, so that $f_m | f_{m+1} + f_{m-1}$. Next, we assume $f_m | f_{m+c} + f_{m-c}$ for $1 \leq c \leq d$ for some value $d < m-1$. Since

$$f_{m+d+1} = 6f_{m+d} - f_{m+d-1} \quad \text{and} \quad f_{m-d-1} = 6f_{m-d} - f_{m-d+1},$$

we know

$$\begin{aligned} f_{m+d+1} + f_{m-(d+1)} &= 6f_{m+d} - f_{m+d-1} + 6f_{m-d} - f_{m-d+1} \\ &= 6(f_{m+d} + f_{m-d}) - (f_{m+d-1} + f_{m-(d-1)}). \end{aligned}$$

By the induction hypothesis, the result follows.

Proposition 2.3. For all $m \geq 1$, $f_m | s_{2m-1}$.

Proof. We see that

$$s_{2m-1} = \sum_{i=1}^{2m-1} (-1)^{i-1} f_i = f_1 - f_2 + f_3 - \cdots + (-1)^{m-1} f_m + \cdots + f_{2m-1}.$$

Notice that this sum is centered about f_m, which divides itself, and that the rest of the terms can be paired in such a way that Proposition 2.2 can be applied easily.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. When $n = 0$, the result is clear. Next, assume $f_m | f_{(2m-1)n+m}$ or $f_m | f_{2mn-n+m}$. We want to prove

$$f_m | f_{(2m-1)(n+1)+m} \quad \text{or} \quad f_m | f_{2mn-n+m+(2m-1)}.$$

Using Proposition 2.1 with $a = 2mn - n + m$, $b = 2m - 1$, and $j = 2m - 2$, we have

$$f_{2mn-n+m+(2m-1)} = s_{2m-1} f_{2mn-n+m+1} - s_{2m-2} f_{2mn-n+m}.$$

From Proposition 2.3, we know $f_m | s_{2m-1}$, and from the induction hypothesis, $f_m | f_{2mn-n+m}$. The result follows.

References

1. E. Barcucci, S. Brunetti, A. Del Lungo, and F. Del Ristoro, "A Combinatorial Interpretation of the Recurrence $f_{n+1} = 6f_n - f_{n-1}$," *Discrete Mathematics,* 190 (1998), 235–240.

2. J. Bonin, L. Shapiro, and R. Simion, "Some q-analogues of the Schröder Numbers Arising From Combinatorial Statistics on Lattice Paths," *J. Statist. Plann. Inference,* 34 (1993), 35–55.

3. M. Newman, D. Shanks, and H. C. Williams, "Simple Groups of Square Order and an Interesting Sequence of Primes," *Acta Arithmetica,* XXXVIII (1980), 129–140.

4. J. Sellers and H. Williams, "On the Infinitude of Composite NSW Numbers," *The Fibonacci Quarterly,* 40 (2002), 253–254.

James A. Sellers
Department of Mathematics
Pennsylvania State University
107 Whitmore Laboratory
University Park, PA 16802
email: sellersj@math.psu.edu