A BOUND FOR THE SQUARE OF THE ZEROS

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Abstract. In this note we use matrix methods for obtaining an explicit bound for the moduli of the square of the zeros of a polynomial.

1. Introduction. It is well-known that matrix methods can be used to obtain certain root-location theorems for polynomial equations. That the zeros of the monic polynomial

$$A_n(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n,$$

are the eigenvalues of its companion matrix leads to easier methods in obtaining bounds for the zeros of polynomials [1,2]. Among others, Aziz and Mohammad [3], by making use of Gershgorin's Theorem [4], have derived some bounds as a direct consequence of the above fact. In this note we apply Gershgorin's result to obtain, as far as we know, a new explicit bound for the squares of the zeros of $A_n(z)$.

Theorem 1. (Gershgorin). Let $A = (a_{ij})$ be an $n \times n$ complex matrix, and let R_i be the sum of the moduli of the off-diagonal elements in the ith row. Then each eigenvalue of A lies in the union of the circles

$$|z - a_{ii}| \le R_i, \quad i = 1, 2, \ldots, n.$$

The analogous result holds if columns of A are considered.

2. The Main Result. In what follows an explicit expression of the coefficients of the polynomials whose zeros are the squares of those of $A_n(z)$ in terms of their coefficients is given, and one theorem on location of the zeros is proved. It can be stated as the following theorem.

Theorem 2. Let $z_1, z_2, \ldots, z_n$ be the zeros of the monic complex polynomial

$$A_n(z) = \sum_{k=0}^{n} a_k z^k.$$

Then, z_i^2, $i = 1, 2, \ldots, n$ lies in the disk $\mathcal{C} = \{z \in \mathbb{C} : |z| \le r\}$, where

$$r = 2 \max_{0 \le k \le n-1} \left\{ \frac{|a_k|^2 + \sum_{j=1}^{n-k+1} |a_{k-j} a_{k+j}|}{|a_{k+1}|^2} \right\}$$

with $a_l = 0$ if $l < 0$ or $l > n$.

Proof. The preceding statement is a consequence of Gershgorin's result. In order to prove it, we will use the larger circles

$$|z| \leq |a_{ii}| + R_i, \quad i = 1, 2, \ldots, n,$$

instead of the ones given in Theorem 1. Let

$$B_n(z) = \sum_{k=0}^{n} b_k z^k$$

be the monic complex polynomial whose zeros are $z_1^2, z_2^2, \ldots, z_n^2$ and let

$$F(B_n) = \begin{pmatrix} 0 & 0 & \cdots & 0 & -b_0 \\ 1 & 0 & \cdots & 0 & -b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -b_{n-1} \end{pmatrix}$$

be its companion matrix. Setting

$$D = \operatorname{diag}(|a_1|^2, |a_2|^2, \ldots, |a_{n-1}|^2, |a_n|^2),$$

we form the matrix

$$D^{-1}FD = \begin{pmatrix} 0 & 0 & \cdots & 0 & -b_0/|a_1|^2 \\ \frac{|a_1|^2}{|a_2|^2} & 0 & \cdots & 0 & -b_1/|a_2|^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{|a_{n-1}|^2}{|a_n|^2} & -b_{n-1}/|a_n|^2 \end{pmatrix}.$$

Since the eigenvalues of $D^{-1}FD$ are the same as those of F, i.e., the zeros of $B_n(z)$; by direct application of Theorem 1, we have

$$|z_i^2| \leq \max_{1 \leq k \leq n-1} \left\{ \frac{|b_0|}{|a_1|^2}, \frac{|a_k|^2}{|a_{k+1}|^2} + \frac{|b_k|}{|a_{k+1}|^2} \right\}, \quad i = 1, 2, \ldots, n. \tag{2.1}$$

On the other hand, the coefficients of $B_n(z)$ are related to coefficients of $A_n(z)$ by the expressions [5]:

$$b_k = (-1)^{n-k}\left(a_k^2 + 2\sum_{j=1}^{n-k+1}(-1)^j a_{k-j}a_{k+j}\right), \quad k = 0, 1, \ldots, n$$

with $a_l = 0$ if $l < 0$ or $l > n$. Substituting (2.2) into (2.1), we have

$$\begin{aligned}
|z_i^2| &\le \max_{1\le k\le n-1}\left\{\frac{|b_0|}{|a_1|^2}, \frac{|a_k|^2 + |b_k|}{|a_{k+1}|^2}\right\} \\
&= \max_{1\le k\le n-1}\left\{\frac{|a_0|^2}{|a_1|^2}, \frac{|a_k|^2 + |(-1)^{n-k}\big(a_k^2 + 2\sum_{j=1}^{n-k+1}(-1)^j a_{k-j}a_{k+j}\big)|}{|a_{k+1}|^2}\right\} \\
&\le \max_{1\le k\le n-1}\left\{\frac{|a_0|^2}{|a_1|^2}, \frac{2|a_k^2| + 2\sum_{j=1}^{n-k+1}|a_{k-j}a_{k+j}|}{|a_{k+1}|^2}\right\} \\
&\le \max_{0\le k\le n-1}\left\{\frac{|a_k^2| + \sum_{j=1}^{n-k+1}|a_{k-j}a_{k+j}|}{|a_{k+1}|^2}\right\}, \quad i = 1, 2, \ldots, n.
\end{aligned}$$

Note that the equality holds only when $a_0 = 0$.

Finally, we have to prove (2.2). We argue by mathematical induction. The first cases when $n = 1, 2, 3$ can be checked by inspection. Next, we assume that the following expression holds

$$B_n(z) = \sum_{k=0}^{n}(-1)^{n-k}\left[a_k^2 + 2\sum_{j=1}^{n-k+1}(-1)^j a_{k-j}a_{k+j}\right]z^k.$$

We have to prove that

$$B_{n+1}(z) = z^{n+1} + \sum_{k=0}^{n}(-1)^{n-k+1}\left[a_k^2 + 2\sum_{j=1}^{n-k+2}(-1)^j a_{k-j}a_{k+j}\right]z^k.$$

In fact,

$$\begin{aligned}
B_{n+1}(z) &= B_n(z)(z - z_{n+1}^2) \\
&= \left\{ \sum_{k=0}^{n} (-1)^{n-k} \left[a_k^2 + 2 \sum_{j=1}^{n-k+1} (-1)^j a_{k-j} a_{k+j} \right] z^k \right\} (z - z_{n+1}^2) \\
&= \sum_{k=0}^{n} (-1)^{n-k} \left[a_k^2 + 2 \sum_{j=1}^{n-k+1} (-1)^j a_{k-j} a_{k+j} \right] z^{k+1} \\
&\quad - z_{n+1}^2 \left\{ \sum_{k=0}^{n} (-1)^{n-k} \left[a_k^2 + 2 \sum_{j=1}^{n-k+1} (-1)^j a_{k-j} a_{k+j} \right] z^k \right\} \\
&= (-1)^{n+1} a_0^2 z_{n+1}^2 + (-1)^n \left[a_0^2 + z_{n+1}^2 (a_1^2 + 2(-1)^1 a_0 a_2) \right] z + \cdots \\
&\quad + (-1)^1 \left[a_{n-1}^2 + 2(-1)^1 a_{n-2} + z_{n+1}^2 \right] z^n + z^{n+1}.
\end{aligned} \tag{2.3}$$

Finally, taking into account that

$$\begin{aligned}
A_{n+1}(z) &= A_n(z)(z - z_{n+1}) = z^{n+1} + \sum_{k=1}^{n} (a_{k-1} - a_k z_{n+1}) z^k - a_0 z_{n+1} \\
&= \sum_{k=0}^{n+1} a_{n+1,k} z^k, \quad a_{n+1,n+1} = 1,
\end{aligned}$$

and (2.3), we have

$$\begin{aligned}
B_{n+1}(z) &= (-1)^{n+1} a_{n+1,0}^2 + (-1)^n [a_{n+1,1}^2 + 2(-1)^1 a_{n+1,0} a_{n+1,2}] z + \cdots \\
&\quad + (-1)^1 [a_{n+1,n}^2 + 2(-1)^1 a_{n+1,n-1}] z^n + z^{n+1} \\
&= z^{n+1} + \sum_{k=0}^{n} (-1)^{n+1-k} \left[a_{n+1,k}^2 + 2 \sum_{j=1}^{n-k+2} (-1)^j a_{n+1,k-j} a_{n+1,k+j} \right] z^k
\end{aligned}$$

and (2.2) is proved. Note that the first subscript in the preceding expressions has been used for pointing out the polynomial degree. This completes the proof of Theorem 2.

For example, if we consider the polynomial

$$A(z) = z^3 - 1.1z^2 - 1.2z + 1.3,$$

the squares of all its zeros lie in the disk $\mathcal{C} = \{z \in \mathbb{C} : |z| < r\}$ where $r \simeq 4.82$. This bound is sharper than the explicit bound of Cauchy $\mathcal{D} = \{z \in \mathbb{C} : |z| < 5.3\}$.

References

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