FUN WITH THE σ(n) FUNCTION

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1. Notation (Standard and Otherwise). The function $\sigma(n)$ is one of the basic number-theoretic arithmetic functions. It is defined as:

$$\sigma(n) = \sum_{d|n} d.$$

Some values of $\sigma(n)$ for small n can be found in [2, sequence A000203]. (Note: It is known that $\sigma(n)$ is also multiplicative, i.e., if j and k have no factors in common other than 1, $\sigma(jk) = \sigma(j)\sigma(k)$.) The Dirichlet convolution of two arithmetic functions $f(n)$ and $g(n)$, itself a function of n, is defined as follows.

$$f * g = \sum_{d|n} f(d)g\left(\frac{n}{d}\right).$$

We will use $\perp$ to denote relative primality.

2. Sigma-Primes. A number n is called *sigma-prime* if and only if $n \perp \sigma(n)$. The sigma-prime numbers below 100 can be found in [2, sequence A014567]. Two rather straightforward theorems are the following.

Theorem 1. All powers of primes are sigma-prime.

Theorem 2. No perfect numbers are sigma-prime.

To build this theory, we shall, in the time-honored tradition of mathematics, start with the simple examples and move up. If a number n is the product of two primes, say p and q, then $\sigma(n) = 1 + p + q + pq = \sigma(p)\sigma(q)$. Now, the only divisors of pq are p and q. Clearly, $p \perp \sigma(p)$. Thus, $p \perp \sigma(pq)$ if and only if $p \perp \sigma(q)$. Similarly, $q \perp \sigma(pq)$ if and only if $q \perp \sigma(p)$. Assuming $p < q$, we can see that (unless $p = 2$ and $q = 3$), $p + 1 < q$ and from that $p + 1 \perp q$. Note also that in the case of the above exception, $q + 1 \not\perp p$. Thus, we can generalize and say that $n = pq$ is sigma-prime if and only if $q + 1 \perp p$. This easily extends to the following theorem.

Theorem 3. If $n = p_1 p_2 \cdots p_k$, where $p_1 < p_2 < \cdots < p_k$, and each of $p_1, \ldots,$ p_k is a prime, then n is sigma-prime if and only if $p_i \perp 1 + p_j$ whenever $i < j$.

This also leads to the following corollary.

Corollary 1. If k is odd and greater than 1, then $2k$ is not sigma-prime.

Proof. Since k is odd, it has at least one odd prime factor p; thus, $2k$ has 2 and p for prime factors. But, $2 \not\perp 1 + p$.

Now we come to the cases where n has prime powers as factors. Thus, if $n = \prod_i {p_i}^{e_i}$, then $\sigma(n) = \prod_i \sigma({p_i}^{e_i})$. Then if n is to be sigma-prime, each of the factors of the first product must be relatively prime to each of the factors of the second product. Unfortunately, the trick we used in the single-power case will not work here; $p_i < p_j$ will not imply that ${p_i}^{e_i} < {p_j}^{e_j}$.

Theorem 4. A number $n = \prod_i {p_i}^{e_i}$ is sigma-prime if and only if for every i, j,

$$p_i \perp \sum_{k=0}^{e_j} {p_j}^k.$$

We close this section with a surprising theorem and a conjecture.

Theorem 5. The square of an even perfect number is sigma-prime.

Proof. According to a famous result of Euler, every even perfect number is of the form $(2^{p-1}(2^p - 1))$, where $2^p - 1$ is prime. Thus, the square of an even perfect number is of the form $(2^{2p-2}(2^p-1)^2)$, where 2 and 2^p-1 are its prime factors. Thus, we have two things to check: (1) $2 \perp 1+(2^p-1)+(2^{2p}-2^{p+1}+1) = 2^{2p}-2^{p+1}+2^p+1$. This is obvious, as the left-hand side is 2 and the right-hand side is odd; and (2) $2^p - 1 \perp 1 + 2 + 2^2 + \cdots + 2^{2p-2} = 2^{2p-1} - 1$. We will prove this by contradiction. Assume that, in fact, $2^p-1|2^{2p-1}-1$. We know that $2^p-1|(2^p-1)^2 = 2^{2p}-2^{p+1}+1$. By our assumption, we also know that $2^p-1|2^{2p}-2$ (by multiplying the right-hand side by 2). Then $2^p - 1$ must divide the (absolute) difference of these two numbers, which is $2^{p+1} - 3$. But this is $2(2^p - 1) - 1$, and this implies that $2^p - 1|1$. This is a contradiction (as $2^p - 1 \geq 3$), and the theorem is proved.

Conjecture. The natural density of the set of sigma-prime numbers is zero.

This seems a reasonable conjecture to make; the set of prime powers has density zero and the set of sigma-prime numbers is not much larger. However, no proof has been forthcoming.

3. Dirichlet Inverse of $\sigma(\mathbf{n})$. To discuss an inverse, we must first have an identity. Looking at the definition of Dirichlet convolution, after a little thought we see that the value of the identity function must be one at $n = 1$ and zero elsewhere.

This tells us immediately that $\sigma^{-1}(1) = 1/\sigma(1) = 1$. It has been shown that the inverse of a multiplicative function is itself multiplicative (for a proof, see [1]), so we need only concern ourselves with the prime powers.

Theorem 6. $\sigma^{-1}(p) = -p - 1$, where p is a prime.

Proof. Since $p > 1$, the identity value under Dirichlet convolution has the value 0 at p. Then, $0 = \sigma(p)\sigma^{-1}(1) + \sigma(1)\sigma^{-1}(p) = (p+1) + \sigma^{-1}(p)$, and therefore, $\sigma^{-1}(p) = -p - 1$.

Theorem 7. $\sigma^{-1}(p^2) = p$, where p is a prime.

Proof. Again, the identity value is 0. Then,

$$\begin{aligned}
0 &= \sigma(p^2)\sigma^{-1}(1) + \sigma(p)\sigma^{-1}(p) + \sigma(1)\sigma^{-1}(p^2) \\
&= (1 + p + p^2) + (p + 1)(-p - 1) + \sigma^{-1}(p^2) \\
&= p^2 + p + 1 - p^2 - 2p - 1 + \sigma^{-1}(p^2) \\
&= -p + \sigma^{-1}(p^2).
\end{aligned}$$

Thus,

$$p = \sigma^{-1}(p^2).$$

Theorem 8. $\sigma^{-1}(p^k) = 0$, where p is a prime, for all integer $k \geq 3$.

Proof. We will prove this using induction, so let's start with $k = 3$.

$$\begin{aligned}
0 &= \sigma(p^3)\sigma^{-1}(1) + \sigma(p^2)\sigma^{-1}(p) + \sigma(p)\sigma^{-1}(p^2) + \sigma(1)\sigma^{-1}(p^3) \\
&= (1 + p + p^2 + p^3) + (1 + p + p^2)(-p - 1) + (1 + p)(p) + \sigma^{-1}(p^3) \\
&= (1 + p + p^2 + p^3) + (-1 - p - p^2 - p^3 - p - p^2) + (p + p^2) + \sigma^{-1}(p^3) \\
&= \sigma^{-1}(p^3).
\end{aligned}$$

That takes care of the base case; let's assume that $\sigma^{-1}(p^n) = 0$ for all n, $3 \le n < k$ and see what happens.

$$\begin{aligned}
0 &= \sigma(p^k)\sigma^{-1}(1) + \sigma(p^{k-1})\sigma^{-1}(p) + \sigma(p^{k-2})\sigma^{-1}(p^2) + \cdots + \sigma(1)\sigma^{-1}(p^k) \\
&= (1 + p + p^2 + \cdots + p^k) + (1 + p + p^2 + \cdots + p^{k-1})(-p-1) \\
&\quad + (1 + p + p^2 + \cdots + p^{k-2})(p) + \sigma^{-1}(p^k) \\
&= (1 + p + p^2 + \cdots + p^k) - (1 + 2p + 2p^2 + \cdots + 2p^{k-1} + p^k) \\
&\quad + (p + p^2 + \cdots + p^{k-1}) + \sigma^{-1}(p^k) \\
&= \sigma^{-1}(p^k).
\end{aligned}$$

These are all the cases; thus, we can generate the sigma inverse function for all n. If we write n in canonical prime factorization form, $n = \prod_i {p_i}^{e_i}$, and define the sequence $\{a_i\}$ as

$$a_i = \begin{cases} -p-1, & \text{if } e_1 = 1 \\ p, & \text{if } e_1 = 2 \\ 0, & \text{if } e_i > 2 \end{cases}.$$

Then, $\sigma^{-1}(n) = \prod_i a_i$.

Note. The values $\sigma^{-1}(n)$, starting $1, -3, -4, 2, -6, 12, \ldots$, have now been added to Sloane's *Encyclopedia* [2, sequence A046692].

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