THE DIMENSION OF INTERSECTION OF k SUBSPACES

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Abstract. This note presents a formula for expressing the dimension of intersection of k subspaces in an n-dimensional vector space over an arbitrary field $\mathcal{F}$.

It is a well-known fundamental formula in linear algebra that for any two subspaces V_1 and V_2 in an m-dimensional vector space V, the dimension of the intersection $V_1 \cap V_2$ is

$$\dim(V_1 \cap V_2) = \dim V_1 + \dim V_2 - \dim(V_1 + V_2). \tag{1}$$

Written in matrix form, it is equivalent to

$$\dim[R(A_1) \cap R(A_2)] = \mathrm{rk}(A_1) + \mathrm{rk}(A_2) - \mathrm{rk}[A_1, A_2], \tag{2}$$

where $A_1 \in \mathcal{F}^{m \times n_1}$, $A_2 \in \mathcal{F}^{m \times n_2}$, $R(\cdot)$ stands for the range (column space) of a matrix.

In this note we extend it to the k-term case to establish a formula for calculating the dimension of the intersection of k column spaces $R(A_1), R(A_2), \ldots, R(A_k)$ using ranks and generalized inverses of matrices.

Let A be an $m \times n$ matrix over an arbitrary field $\mathcal{F}$. A generalized inverse of A is defined as a solution of the matrix equation $AXA = A$ and often denoted by A^-. For the homogeneous matrix equation $AX = 0$, where $X \in \mathcal{F}^{n \times p}$, its general solution can be written as $X = (I_n - A^-A)U$, where $U \in \mathcal{F}^{n \times p}$ is arbitrary [2]. Two basic rank equalities used in the sequel related to generalized inverses [1] are

$$\mathrm{rk}[A, B] = \mathrm{rk}(A) + \mathrm{rk}(B - AA^-B), \tag{3}$$

and

$$\mathrm{rk}\begin{bmatrix} A \\ C \end{bmatrix} = \mathrm{rk}(A) + \mathrm{rk}(C - CA^-A). \tag{4}$$

Our main result is given below.

Theorem. Let $A_i \in \mathcal{F}^{m\times n_i}$, $i = 1, 2, \ldots, k$. Then the dimension of the intersection $\cap_{i=1}^{k} R(A_i)$ is

$$\dim\left(\bigcap_{i=1}^{k} R(A_i)\right) = \mathrm{rk}(A_1) + \mathrm{rk}(A_2) + \cdots + \mathrm{rk}(A_k) - \mathrm{rk}\begin{bmatrix} A_1 & A_2 & & & \\ A_1 & & A_3 & & \\ \vdots & & & \ddots & \\ A_1 & & & & A_k \end{bmatrix}. \tag{5}$$

Proof. In order to prove (5), we first find a general expression of vectors in $\cap_{i=1}^{k} R(A_i)$. Let $X \in \cap_{i=1}^{k} R(A_i)$. Then it is obvious that there must exist $X_i \in \mathcal{F}^{n_i\times 1}$ such that

$$X = A_1X_1 = A_2X_2 = \cdots = A_kX_k. \tag{6}$$

Consider it as a system of linear matrix equations, we can write it as

$$\begin{bmatrix} I_m & -A_1 & & & \\ I_m & & -A_2 & & \\ \vdots & & & \ddots & \\ I_m & & & & -A_k \end{bmatrix} \begin{bmatrix} X \\ X_1 \\ \vdots \\ X_k \end{bmatrix} = 0, \tag{7}$$

or simply $MY = 0$. Solving for Y, we then get $Y = (I_t - M^-M)U$, where $U \in \mathcal{F}^{t\times 1}$, $t = m + n_1 + \cdots + n_k$. In that case, the general expression of X is

$$X = [I_m, 0, \cdots, 0]Y = [I_m, 0, \cdots, 0](I_t - M^-M)U.$$

Thus the dimension of $\cap_{i=1}^{k} R(A_i)$, by (4) is

$$\dim\Big(\bigcap_{i=1}^{k} R(A_i)\Big) = \mathrm{rk}[I_m, 0, \cdots, 0](I_t - M^{-}M)$$

$$= \mathrm{rk}\begin{bmatrix} I_m & 0 & \cdots & 0 \\ I_m & -A_1 & & \\ \vdots & & \ddots & \\ I_m & & & -A_k \end{bmatrix} - \mathrm{rk}\begin{bmatrix} I_m & -A_1 & & \\ \vdots & & \ddots & \\ I_m & & & -A_k \end{bmatrix}$$

$$= m + \mathrm{rk}\begin{bmatrix} -A_1 & & \\ & \ddots & \\ & & -A_k \end{bmatrix} - m - \mathrm{rk}\begin{bmatrix} A_1 & -A_2 & & & \\ A_1 & & -A_3 & & \\ \vdots & & & \ddots & \\ A_1 & & & & -A_k \end{bmatrix}$$

$$= \mathrm{rk}(A_1) + \mathrm{rk}(A_2) + \cdots + \mathrm{rk}(A_k) - \mathrm{rk}\begin{bmatrix} A_1 & A_2 & & & \\ A_1 & & A_3 & & \\ \vdots & & & \ddots & \\ A_1 & & & & A_k \end{bmatrix},$$

establishing (5).

Corollary. Let $A_i \in \mathcal{F}^{m\times n_i}$, $i = 1, 2, \ldots, k$. Then $\cap_{i=1}^{k} R(A_i) = \{0\}$ holds if and only if

$$\mathrm{rk}\begin{bmatrix} A_1 & A_2 & & & \\ A_1 & & A_3 & & \\ \vdots & & & \ddots & \\ A_1 & & & & A_k \end{bmatrix} = \mathrm{rk}(A_1) + \mathrm{rk}(A_2) + \cdots + \mathrm{rk}(A_k). \tag{8}$$

References

1. G. Marsaglia and G. P. H. Styan, "Equalities and Inequalities for Ranks of Matrices," *Linear and Multilinear Algebra*, 2 (1974), 269–292.

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