NOTE ON BOUNDS OF THE ZEROS

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Abstract. In this note we give two new explicit bounds for the moduli of the zeros involving binomial coefficients and Fibonacci's numbers.

1. Introduction. Since the time of Gauss and Cauchy many papers devoted to giving bounds for the zeros of polynomials have appeared. In some of them new bounds were discovered, in others classical bounds were improved. Since the beginning, binomial coefficients have appeared in the derivation or as part of closed expressions of bounds [1]. However, as far as we know, Fibonacci's numbers (i.e., $F_0 = 0, F_1 = 1$ and for $n \geq 2, F_n = F_{n-1} + F_{n-2}$) have never appeared either in implicit bounds or explicit bounds for the moduli of the zeros. In this paper we determine in the complex plane circular domains containing all the zeros of a polynomial where binomial coefficients and Fibonacci's numbers appear.

2. The Main Result. In what follows two numerical identities are considered and one theorem on location of the zeros is proved.

Theorem 2.1. Let $\underset{\cdot}{A}(z) = \sum_{k=0}^{n} a_k z^k$ be a complex monic polynomial. Then, all its zeros lie in the disks $\mathcal{C}_1 = \{z \in \mathbb{C} : |z| \leq r_1\}$ or $\mathcal{C}_2 = \{z \in \mathbb{C} : |z| \leq r_2\}$, where

$$r_1 = \max_{1 \leq k \leq n} \left\{ \sqrt[k]{\frac{2^{n-1} C(n+1,2)}{k^2 C(n,k)} |a_{n-k}|} \right\}, \tag{2.1}$$

and

$$r_2 = \max_{1 \leq k \leq n} \left\{ \sqrt[k]{\frac{F_{3n}}{C(n,k) 2^k F_k} |a_{n-k}|} \right\}. \tag{2.2}$$

Proof. In order to prove the above statement we consider the following numerical identities:

$$\sum_{k=1}^{n} k^2 C(n,k) = 2^{n-2} n(n+1) \tag{2.3}$$

$$\sum_{k=0}^{n} C(n,k)2^k F_k = F_{3n}. \tag{2.4}$$

It is known [2] that all the zeros of $A(z)$ have modulus less than or equal to ξ, the unique positive root of the equation

$$B(z) = z^n - |a_{n-1}|z^{n-1} - \cdots - |a_1|z - |a_0| = 0.$$

Therefore, our statement will be proved if we show that $r_j \geq \xi, j = 1,2$ or equivalently if we prove that $B(r_j) \geq 0, j = 1,2$.

From (2.1), we have

$$|a_{n-k}| \leq \frac{k^2 C(n,k)}{2^{n-1}C(n+1,2)}\, r_1^k,\ k = 1,2,\cdots,n.$$

Thus,

$$B(r_1) = r_1^n - \sum_{k=1}^{n} |a_{n-k}|r_1^{n-k} \geq r_1^n - \sum_{k=1}^{n} \Big\{\frac{k^2 C(n,k)}{2^{n-1}C(n+1,2)} r_1^k\Big\} r_1^{n-k}$$

$$= r_1^n \Big(1 - \sum_{k=1}^{n} \frac{k^2 C(n,k)}{2^{n-1}C(n+1,2)}\Big) = 0,$$

and we are done.

From (2.2), we have

$$|a_{n-k}| \leq \frac{C(n,k)2^k F_k}{F_{3n}}\, r_2^k,\ k = 1,2,\cdots,n.$$

Then,

$$B(r_2) = r_2^n - \sum_{k=1}^{n} |a_{n-k}|r_2^{n-k} \geq r_2^n - \sum_{k=1}^{n} \Big\{\frac{C(n,k)2^k F_k}{F_{3n}} r_2^k\Big\} r_2^{n-k}$$

$$= r_2^n\Big(1 - \sum_{k=1}^{n} \frac{C(n,k)2^k F_k}{F_{3n}}\Big) = 0,$$

and the second part is proved.

Finally, we should establish (2.3) and (2.4). Identity (2.3) can be easily proved by induction or by applying two times the operator $\Big(z\dfrac{d}{dz}\Big)$ to

$$(1+z)^n = \sum_{k=0}^{n} C(n,k)z^k.$$

In fact,

$$\Big(z\frac{d}{dz}\Big)\Big\{(1+z)^n\Big\} = \Big(z\frac{d}{dz}\Big)\Big\{\sum_{k=0}^{n} C(n,k)z^k\Big\},$$

$$nz(1+z)^{n-1} = \sum_{k=1}^{n} kC(n,k)z^k.$$

Applying newly $\Big(z\dfrac{d}{dz}\Big)$, we get

$$nz(1+z)^{n-1} + n(n-1)z^2(1+z)^{n-2} = \sum_{k=1}^{n} k^2C(n,k)z^k. \tag{2.5}$$

Now, we take $z = 1$ in (2.5) and identity (2.3) is proved. To prove identity (2.4), see [3, 4].

For example, if we consider the polynomial $A(z) = z^3 + 0.1z^2 + 0.5z + 0.7$, it has all its zeros in the disk $\mathcal{C}_1 = \{z \in \mathbb{C} : |z| \le r_1\}$ or $\mathcal{C}_2 = \{z \in \mathbb{C} : |z| \le r_2\}$, where $r_1 \simeq 1.23$ and $r_2 \simeq 1.19$. In both cases, these bounds are sharper than the explicit bound of Cauchy $|z| < 1.7$.

References

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