SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

26. [1990, 140; 1991, 152] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Prove that

$$\sum_{k=1}^{38} \sin \frac{k^8 \pi}{38} = \sqrt{19}.$$

Solution by Les Reid, Southwest Missouri State University, Springfield, Missouri. We will show more generally that if p is a prime number, $p \equiv 3 \mod 4$, and $m \geq 1$, then

$$\sum_{k=1}^{2p} \sin \frac{k^{2^m} \pi}{2p} = \sum_{k=1}^{2p} \cos \frac{k^{2^m} \pi}{2p} = \sqrt{p}.$$

Let $\omega = \cos \pi/(2p) + i \sin \pi/(2p)$ and consider $\sum_{k=1}^{4p} \omega^{k^{2^m}}$. Since ω is a primitive $4p^{th}$ root of unity, it suffices to compute the exponents modulo $4p$.

We claim that $\sum_{k=1}^{4p} \omega^{k^{2^m}}$ is independent of m for $m \geq 1$. To show this it suffices to show that $\{k^2 | k \in \mathbb{Z}_{4p}\} = \{k^4 | k \in \mathbb{Z}_{4p}\}$ and the claim will follow by induction. Since $\{k^4 | k \in \mathbb{Z}_{4p}\} \subseteq \{k^2 | k \in \mathbb{Z}_{4p}\}$, we will be done if we can construct a bijection from the superset to the subset. We claim that the map $f(x) = x^2$ does the trick. Now $\mathbb{Z}_{4p} \cong \mathbb{Z}_4 \times \mathbb{Z}_p$, by the Chinese Remainder Theorem. There are four types of squares in $\mathbb{Z}_4 \times \mathbb{Z}_p$: $(0,0)$, $(1,0)$, $(0,u^2)$, and $(1,u^2)$ where $u \neq 0 \in \mathbb{Z}_p$. Squaring clearly leaves the first two types invariant. For the second two types, the first coordinate is unchanged by squaring. The second coordinate is from the multiplicative abelian group $(\mathbb{Z}_p^{\times})^2$ which has order $(p-1)/2$ (since p is an (odd) prime). This is odd since $p \equiv 3 \mod 4$. Therefore squaring yields a group isomorphism (on the second coordinate), and hence f is a bijection.

A classic result of Gauss [for example, see Lang's *Algebraic Number Theory*, pp.85-87] states that if $\alpha = \cos 2\pi/b + i \sin 2\pi/b$ for $b > 0$, then

$$\sum_{k=1}^{b} \alpha^{k^2} = \begin{cases} (1+i)\sqrt{b} & \text{if } b \equiv 0 \mod 4 \\ \sqrt{b} & \text{if } b \equiv 1 \mod 4 \\ 0 & \text{if } b \equiv 2 \mod 4 \\ i\sqrt{b} & \text{if } b \equiv 3 \mod 4 \end{cases}$$

In our case, $b = 4p$, so $\sum_{k=1}^{4p} \omega^{k^2} = (1+i)\sqrt{4p}$. Finally $\sum_{k=1}^{4p} \omega^{k^{2^m}} = 2\sum_{k=1}^{2p} \omega^{k^{2^m}}$, so

$$\sum_{k=1}^{2p} (\cos \frac{k^{2^m}\pi}{2p} + i \sin \frac{k^{2^m}\pi}{2p}) = \sum_{k=1}^{2p} \omega^{k^{2^m}} = \frac{1}{2} \sum_{k=1}^{4p} \omega^{k^{2^m}}$$

$$= \frac{1}{2} \sum_{k=1}^{4p} \omega^{k^2} = \frac{1}{2}(1+i)\sqrt{4p} = (1+i)\sqrt{p}.$$

Comparing real and imaginary parts, the result follows.