INTEGRATION OF RATIONAL FUNCTIONS BY THE SUBSTITUTION $\mathbf{x = u^{-1}}$

Joseph Wiener

Integration of a proper rational function by the method of partial fractions usually involves tedious calculations when its denominator contains repeated linear or quadratic factors. Often this difficulty can be alleviated by the substitution $x = u^{-1}$ which is illustrated in the following examples.

Example 1. Integrating the function $R(x) = (x^4 - x^3)^{-1}$ by the substitution $x = u^{-1}$ changes it to $R_1(u) = u^4/(1-u)$ and shows that

$$\int \frac{dx}{x^4 - x^3} = \int \frac{u^4}{1-u} \cdot \frac{du}{-u^2} = \int \frac{u^2}{u-1} du$$

$$= \int \left(u + 1 + \frac{1}{u-1} \right) du = \frac{u^2}{2} + u + \ln|u-1| + C.$$

The answer is

$$\int \frac{dx}{x^4 - x^3} = \frac{1}{2x^2} + \frac{1}{x} + \ln\left| \frac{x-1}{x} \right| + C.$$

Example 2. The denominator of the function $R(x) = (3x-1)/x^4(x-1)^2$ contains two repeated linear factors, and in this case the substitution $x = u^{-1}$ especially enhances integration giving

$$\int \frac{(3x-1)dx}{x^4(x-1)^2} = \int \frac{(3u^3 - u^4)u^2}{(u-1)^2} \cdot \frac{du}{-u^2}$$

$$= \int \frac{u^3(u-3)}{(u-1)^2} du.$$

Now it is useful to make the substitution $u = w + 1$ leading to the result

$$\int \frac{(w+1)^3(w-2)}{w^2}dw = \int \frac{w^4 + w^3 - 3w^2 - 5w - 2}{w^2}dw.$$

Then term by term division and integration yields

$$\begin{aligned}\int R(x)dx &= \frac{1}{3}w^3 + \frac{1}{2}w^2 - 3w - 5\ln|w| + \frac{2}{w} + C \\ &= -\frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + 3\left(\frac{x-1}{x}\right) - 2\left(\frac{x}{x-1}\right) \\ &\quad - 5\ln\left|\frac{x-1}{x}\right| + C.\end{aligned}$$

Example 3. The substitution $x = u^{-1}$ transforms the integral of $R(x) = 1/x^6(x^2+1)$ into

$$\begin{aligned}\int \frac{dx}{x^6(x^2+1)} &= \int \frac{u^8}{u^2+1} \cdot \frac{du}{-u^2} = -\int \frac{u^6}{u^2+1}du \\ &= -\int \left(u^4 - u^2 + 1 - \frac{1}{u^2+1}\right)du \\ &= -\frac{1}{5}u^5 + \frac{1}{3}u^3 - u + \tan^{-1} u + C \\ &= -\frac{1}{5x^5} + \frac{1}{3x^3} - \frac{1}{x} - \tan^{-1} x + C.\end{aligned}$$

Example 4. For the improper integral

$$I = \int_0^\infty \frac{dx}{(x^2+1)^2},$$

the substitution $x = u^{-1}$ gives

$$I = \int_\infty^0 \frac{u^4}{(u^2+1)^2} \cdot \frac{du}{-u^2} = \int_0^\infty \frac{u^2 du}{(u^2+1)^2} = \int_0^\infty \frac{x^2 dx}{(x^2+1)^2}.$$

Therefore,

$$2I = \int \frac{1}{(x^2+1)^2} dx + \int \frac{x^2}{(x^2+1)^2} dx = \int \frac{dx}{x^2+1} = \frac{\pi}{2},$$

whence $I = \pi/4$.

Example 5. The substitution $x = u^{-1}$ effectively tames the challenging improper integral

$$J = \int_0^\infty \frac{dx}{x^4+1}$$

changing it to the form

$$J = \int_\infty^0 \frac{u^4}{u^4+1} \cdot \frac{du}{-u^2} = \int_0^\infty \frac{x^2}{x^4+1} dx.$$

Adding both equations for J,

$$2J = \int_0^\infty \frac{(x^2+1)dx}{x^4+1},$$

and dividing each term in the numerator and denominator by x^2 generates the integral

$$\int_0^\infty \frac{(1+x^{-2})dx}{x^2+x^{-2}} = \int_0^\infty \frac{d(x-x^{-1})}{(x-x^{-1})^2+2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{(x-x^{-1})}{\sqrt{2}} \Bigg|_0^\infty = \frac{\pi}{\sqrt{2}}.$$

Hence, $J = \pi\sqrt{2}/4$, and it is interesting to note that this integral is used in the computation of Fresnel's integrals

$$\int_0^\infty \sin\big(x^2\big)dx = \int_0^\infty \cos\big(x^2\big)dx = \sqrt{2\pi}/4.$$

Also, the substitution $x = \sqrt{\tan\theta}$ shows that

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{\tan\theta}} = 2\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}.$$

The integral on the left reappears from time to time in problem sections of some journals.

Example 6. Consider the integral

$$F(x) = \int \frac{dx}{(1-x^2)^{n+1}} = \int \frac{dx}{(1-x)^{n+1}(1+x)^{n+1}}$$

and set $1 - x = y$ giving

$$F = -\int \frac{dy}{y^{n+1}(2-y)^{n+1}}.$$

Next, the substitution $y = u^{-1}$ changes this integral to

$$F = \int \frac{u^{2n}}{(2u-1)^{n+1}}du,$$

and the substitution $t = 2u - 1$ leads to the integral

$$F = \frac{1}{2^{2n+1}}\int \frac{(t+1)^{2n}}{t^{n+1}}dt.$$

By the binomial formula,

$$F = \frac{1}{2^{2n+1}} \sum_{i=0}^{2n} \binom{2n}{i} \int t^{n-i-1} dt,$$

and

$$F = \frac{1}{2^{2n+1}} \left[\binom{2n}{n} \ln |t| + \sum_{i=0}^{2n} \binom{2n}{i} \frac{t^{n-i}}{n-i} \right] + C, \quad (i \neq n).$$

Since $t = (1+x)/(1-x)$, the original integral becomes

$$F(x) = \frac{1}{2^{2n+1}} \left[\binom{2n}{n} \ln \left| \frac{1+x}{1-x} \right| + \sum_{i=0}^{2n} \binom{2n}{i} \frac{1}{n-i} \left(\frac{1+x}{1-x} \right)^{n-i} \right] + C, \quad (i \neq n).$$

This formula can also be simplified by writing

$$F(x) = \frac{1}{2^{2n+1}} \left[\binom{2n}{n} \ln \left| \frac{1+x}{1-x} \right| + \sum_{i=0}^{n-1} \binom{2n}{i} \frac{1}{n-i} \left(\frac{1+x}{1-x} \right)^{n-i} \right.$$
$$\left. - \sum_{i=n+1}^{2n} \binom{2n}{i} \frac{1}{i-n} \left(\frac{1-x}{1+x} \right)^{i-n} \right] + C$$

and changing i to $2n-i$ in the second sum. Then

$$F(x) = \frac{1}{2^{2n+1}} \binom{2n}{n} \ln \left| \frac{1+x}{1-x} \right|$$
$$+ \frac{1}{2^{2n+1}} \sum_{i=0}^{n-1} \binom{2n}{i} \frac{1}{n-i} \left[\left(\frac{1+x}{1-x} \right)^{n-i} - \left(\frac{1-x}{1+x} \right)^{n-i} \right] + C,$$

or

$$F(x) = \frac{1}{2^{2n+1}}\binom{2n}{n}\ln\left|\frac{1+x}{1-x}\right|$$

$$+\frac{1}{2^{2n+1}}\sum_{i=1}^{n}\binom{2n}{n-i}\frac{1}{i}\left[\left(\frac{1+x}{1-x}\right)^{i}-\left(\frac{1-x}{1+x}\right)^{i}\right]+C.$$

Example 7. The substitution $x = \sin\theta$ in the preceding integral yields a remarkable closed formula for the integral

$$\int \sec^{2n+1}\theta d\theta = \frac{1}{2^{2n}}\binom{2n}{n}\ln|\sec\theta+\tan\theta|$$

$$+\frac{1}{2^{2n+1}}\sum_{i=1}^{n}\binom{2n}{n-i}\frac{1}{i}\Big[(\sec\theta+\tan\theta)^{2i}$$

$$-(\sec\theta-\tan\theta)^{2i}\Big]+C.$$

Similarly, the substitution $x = \cos\theta$ leads to the closed formula for the integral

$$\int \csc^{2n+1}\theta d\theta = \frac{1}{2^{2n}}\binom{2n}{n}\ln|\csc\theta-\cot\theta|$$

$$+\frac{1}{2^{2n+1}}\sum_{i=1}^{n}\binom{2n}{n-i}\frac{1}{i}\Big[(\csc\theta-\cot\theta)^{2i}$$

$$-(\csc\theta+\cot\theta)^{2i}\Big]+C.$$

Most texts approach these integrals via reduction formulas [1]. The method of differentiating indefinite integrals with respect to a parameter was used in [2] to derive equivalent closed formulas.

Example 8. The substitution $x = u^{-1}$ is very useful in the study of improper integrals since it transforms integrals over $[0, 1]$ and $[1, \infty)$ into each other. Moreover, it changes some improper integrals into proper (definite) integrals. For instance:

$$\int_1^\infty \frac{1}{x^2} dx = \int_1^0 u^2 \cdot \frac{du}{-u^2} = \int_0^1 du; \tag{1}$$

$$\int_1^\infty \frac{dx}{x^2+1} = \int_1^0 \frac{u^2}{1+u^2} \cdot \frac{du}{-u^2} = \int_0^1 \frac{du}{1+u^2}; \tag{2}$$

$$\int_1^\infty \frac{dx}{x^4+1} = \int_1^0 \frac{u^4}{1+u^4} \cdot \frac{du}{-u^2} = \int_0^1 \frac{u^2 du}{1+u^4}. \tag{3}$$

More generally, consider the improper integral over $[1, \infty)$ of a rational function $R(x) = P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials of degrees p and q, respectively, and $Q(x) \neq 0$. The substitution $x = u^{-1}$ changes this integral to

$$\int_1^\infty R(x)dx = \int_1^0 \frac{u^q P_1(u)}{u^p Q_1(u)} \cdot \frac{du}{-u^2} = \int_0^1 u^{q-p-2} \frac{P_1(u)}{Q_1(u)} du,$$

where $P_1(u)$ and $Q_1(u)$ are polynomials of degrees p and q, respectively, and $Q_1(u) \neq 0$ for $u \in [0, 1]$. Now it is seen that the latter integral has no singularity if $q \geq p + 2$, which is precisely the condition for convergence of the original integral.

The technique described may be viewed as a useful supplement (not a replacement) to the method of partial fractions which often suggests an alternative approach to related problems, with considerable reduction in the volume of computations.

References

1. H. Anton, *Calculus*, 6th ed., John Wiley & Sons, New York, 1999.

2. J. Wiener, D. Skow, and W. Watkins, "Integrating Powers of Trigonometric Functions," *Missouri Journal of Mathematical Sciences*, 4 (1992), 55-61.

Joseph Wiener
Department of Mathematics
University of Texas – Pan American
Edinburg, TX 78539
email: jwiener@panam.edu