A REPRESENTATION AND SOME PROPERTIES FOR k-FIBONACCI SEQUENCES

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Abstract. The k-Fibonacci sequence $\{g_n^{(k)}\}$ is defined as:

$$g_1^{(k)} = \ldots = g_{k-2}^{(k)} = 0, \;\; g_{k-1}^{(k)} = g_k^{(k)} = 1$$

and for $n > k \geq 2$,

$$g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}.$$

In this paper, we give a combinatorial representation of $g_n^{(k)}$ and give some properties for k-Fibonacci sequence.

1. Introduction. The well-known Fibonacci sequence $\{F_n\}$ is defined as:

$$F_1 = F_2 = 1 \quad \text{and, for} \quad n > 2, \quad F_n = F_{n-1} + F_{n-2}.$$

We call F_n the nth Fibonacci number. The Fibonacci sequence is

$$(F_0 := 0),\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144, \ldots .$$

Now, we consider the generalization of the Fibonacci sequence, which is called the k-Fibonacci sequence for the positive integer $k \geq 2$. The k-Fibonacci sequence $\{g_n^{(k)}\}$ is defined as:

$$g_1^{(k)} = \cdots = g_{k-2}^{(k)} = 0, \;\; g_{k-1}^{(k)} = g_k^{(k)} = 1$$

and for $n > k \geq 2$,

$$g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}.$$

We call $g_n^{(k)}$ the nth k-Fibonacci number. For example, if $k = 2$, then $\{g_n^{(2)}\}$ is the Fibonacci sequence, $\{F_n\}$, and if $k = 4$, then $g_1^{(4)} = g_2^{(4)} = 0$, $g_3^{(4)} = g_4^{(4)} = 1$, and then the 4-Fibonacci sequence is

$$0,\ 0,\ 1,\ 1,\ 2,\ 4,\ 8,\ 15,\ 29,\ 56,\ 108,\ 208,\ 401,\ 773, \ldots .$$

Let I_{k-1} be the identity matrix of order $k-1$ and let E be an $1 \times (k-1)$ matrix whose entries are ones. For any $k \geq 2$, the fundamental recurrence relation

$$g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}$$

can be defined by the vector recurrence relation

$$\begin{bmatrix} g_{n+1}^{(k)} \\ g_{n+2}^{(k)} \\ \vdots \\ g_{n+k}^{(k)} \end{bmatrix} = Q_k \begin{bmatrix} g_n^{(k)} \\ g_{n+1}^{(k)} \\ \vdots \\ g_{n+k-1}^{(k)} \end{bmatrix} \tag{1.1}$$

where

$$Q_k = \begin{bmatrix} 0 & I_{k-1} \\ 1 & E \end{bmatrix}_{k \times k}. \tag{1.2}$$

The matrix Q_k is said to be the k-Fibonacci matrix. By applying (1.1), we have

$$\begin{bmatrix} g_{n+1}^{(k)} \\ g_{n+2}^{(k)} \\ \vdots \\ g_{n+k}^{(k)} \end{bmatrix} = Q_k^n \begin{bmatrix} g_1^{(k)} \\ g_2^{(k)} \\ \vdots \\ g_k^{(k)} \end{bmatrix}.$$

Let $\{g_n^{(k)}\}$ be a k-Fibonacci sequence, and let

$$G_k = (g_1,\ g_2,\ g_3,\ \dots),\ \ g_i = g_{i+k-2}^{(k)}, \quad i = 1,\ 2,\ \dots\ ,$$

and if $i \leq 0$, then $g_i = 0$.

For example, if $k = 2$, then $G_2 = (1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13, \dots)$. And if $k = 4$, then $G_4 = (1,\ 1,\ 2,\ 4,\ 8,\ 15,\ 29,\ 56,\ 108, \dots)$.

In [3], the author considered the completeness on $\{g_n^{(k)}\}$ and gave a representation for the recurrence relation $g_n^{(k)}$. In [4], the authors found a relationship between the k-Fibonacci number $g_n^{(k)}$ and the number of 1-factors of a bipartite graph, and in [5], the authors considered the eigenvalues of k-Fibonacci matrix Q_k and gave some interesting examples in combinatorics and probability with respect to the k-Fibonacci sequences.

In this paper, we give a combinatorial representation of $g_n^{(k)}$ and introduce some properties for k-Fibonacci sequences.

2. Combinatorial representation of $\mathbf{g_n}$. In this section, we give a representation for the nth k-Fibonacci number by using the generating function $G_k(x)$.

We can easily find the characteristic polynomial, $x^k - x^{k-1} - \cdots - x - 1$, of the k-Fibonacci matrix Q_k. It follows that all of the eigenvalues of Q_k satisfy

$$x^k = x^{k-1} + x^{k-2} + \cdots + x + 1.$$

And we can find the following fact in [5]:

$$\begin{aligned} x^n = {} & g_{n-k+2}x^{k-1} + (g_{n-k+1} + g_{n-k} + \cdots + g_{n-2k+3})x^{k-2} \\ & + (g_{n-k+1} + g_{n-k} + \cdots + g_{n-2k+4})x^{k-3} \\ & + \cdots + (g_{n-k+1} + g_{n-k})x + g_{n-k+1}. \end{aligned} \tag{2.1}$$

Let

$$G_k(x) = g_1 + g_2x + g_3x^2 + \cdots + g_{n+1}x^n + \cdots .$$

Then

$$G_k(x) - xG_k(x) - x^2G_k(x) - \cdots - x^kG_k(x) = (1 - x - x^2 - \cdots - x^k)G_k(x).$$

Using equation (2.1), we have

$$(1 - x - x^2 - \cdots - x^k)G_k(x) = g_1 = 1.$$

Thus,

$$G_k(x) = (1 - x - x^2 - \cdots - x^k)^{-1}$$

for $0 \le x + x^2 + \cdots + x^k < 1$.

Let $f_k(x) = x + x^2 + \cdots + x^k$. Then $0 \le f_k(x) < 1$ and we have the following lemma.

Lemma 2.1. For positive integers p and n, the coefficient of x^n in $(f_k(x))^p$ is

$$\sum_{l=0}^{p}(-1)^l\binom{p}{l}\binom{n-kl-1}{n-kl-p}, \quad \frac{n}{k}\le p\le n.$$

Proof.

$$\begin{aligned}(f_k(x))^p &= (x+x^2+\cdots+x^k)^p\\ &= x^p(1+x+x^2+\cdots+x^{k-1})^p\\ &= x^p\left(\frac{1-x^k}{1-x}\right)^p\\ &= x^p\left((1-x^k)\left(\frac{1}{1-x}\right)\right)^p\\ &= x^p\left(\left(\sum_{l=0}^{p}\binom{p}{l}(-1)^l x^{kl}\right)\left(\sum_{i=0}^{\infty}\binom{p+i-1}{i}x^i\right)\right).\end{aligned}$$

In the above equation, we only consider the coefficient of x^n. Since the first term on the right is x^p, $kl+i=n-p$, that is, $i=n-kl-p$. If $l=q$ for any $q=0,1,\dots,p$, then the second term on the right is

$$\left((-1)^q\binom{p}{q}\binom{n-kq-1}{n-kq-p}\right)x^{n-p}.$$

So, the coefficient of x^n is

$$\sum_{l=0}^{p}(-1)^l\binom{p}{l}\binom{n-kl-1}{n-kl-p}, \quad \frac{n}{k}\le p\le n.$$

The proof is completed.

Now we have a combinatorial representation for g_n.

Theorem 2.2. For positive integers p and n,

$$g_{n+1} = \sum_{\frac{n}{k} \le p \le n} \sum_{l=0}^{p} (-1)^l \binom{p}{l} \binom{n-kl-1}{n-kl-p}. \tag{2.2}$$

Proof. Since

$$G_k(x) = g_1 + g_2 x + g_3 x^2 + \cdots + g_{n+1} x^n + \cdots$$

$$= \frac{1}{1 - x - x^2 - \cdots - x^k},$$

the coefficient of x^n is the $n+1$st Fibonacci number, g_{n+1}, in G_k. And,

$$G_k(x) = \frac{1}{1 - x - x^2 - \cdots - x^k}$$

$$= \frac{1}{1 - f_k(x)}$$

$$= 1 + f_k(x) + (f_k(x))^2 + \cdots + (f_k(x))^n + \cdots \tag{2.3}$$

$$= 1 + f_k(x) + x^2 \sum_{l=0}^{2} \binom{2}{l} (-1)^l x^{kl} \sum_{i=0}^{\infty} \binom{i+1}{i} x^i +$$

$$\cdots + x^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l x^{kl} \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i + \cdots .$$

Since we consider the coefficient of x^n, we only need the first $n+1$ terms on the right. The $(p+1)$st term in (2.3) is

$$x^p \sum_{l=0}^{p} \binom{p}{l} (-1)^l x^{kl} \sum_{i=0}^{\infty} \binom{p+i-1}{i} x^i.$$

So, $kl + i = n - p$, and $\frac{n}{k} \leq p \leq n$. Hence, by Lemma 2.1, we have (2.2).

If $k = 2$, then

$$G_2 = (1, 1, 2, 3, 5, 8, 13, 21, \dots)$$

is the Fibonacci sequence $\{F_n\}$. Since the generating function for $\{F_n\}$ is $G_2(x) = \frac{1}{1-x-x^2}$, and hence,

$$\begin{aligned} G_2(x) &= \frac{1}{1-x(1+x)} \\ &= 1 + x(1+x) + x^2(1+x)^2 + \cdots + x^n(1+x)^n + \cdots. \end{aligned}$$

If the first $n+1$ terms on the right are examined in reverse order, it is seen that the coefficient of x^n in $G_2(x)$ is

$$1 + \binom{n-1}{1} + \binom{n-2}{2} + \cdots \tag{2.4}$$

as asserted. So, we have the following corollary.

Corollary 2.3. Let F_{n+1} be the $(n+1)$st Fibonacci number. Then

$$\begin{aligned} F_{n+1} &= \sum_{i=0} \binom{n-i}{i} \\ &= \sum_{\frac{n}{2} \leq p \leq n} \sum_{l=0}^{p} (-1)^l \binom{p}{l} \binom{n-2l-1}{n-2l-p}. \end{aligned}$$

Proof. By (2.2) and (2.4), the proof is completed.

3. Properties of k-Fibonacci Sequences. In this section, we give some properties for k-Fibonacci sequences. First, we have the following theorem by using vector recurrence relation (1.1).

Theorem 3.1 [3]. For positive integers n and m,

$$\begin{aligned} g_{n+m} = g_n g_{m-(k-1)} &+ (g_n + g_{n-1}) g_{m-(k-2)} + \\ &(g_n + g_{n-1} + g_{n-2}) g_{m-(k-3)} + \cdots \\ &+ (g_n + g_{n-1} + g_{n-2} + \cdots + g_{n-(k-2)}) g_{m-1} + g_{n+1} g_m. \end{aligned}$$

Proof. For G_k, $k \geq 2$, since $g_1 = g_2 = 1$, we can replace the matrix Q_k in (2.2) with

$$Q_k = \begin{bmatrix} 0 & g_1 & 0 & \cdots & 0 \\ 0 & 0 & g_1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & & g_1 \\ g_1 & g_1 & \cdots & g_1 & g_2 \end{bmatrix}.$$

Then

$$Q_k^n = \begin{bmatrix} g_{n-(k-1)} & g^{\dagger}_{1,2} & g^{\dagger}_{1,3} & \cdots & g^{\dagger}_{1,k-1} & g_{n-(k-2)} \\ g_{n-(k-2)} & g^{\dagger}_{2,2} & g^{\dagger}_{2,3} & \cdots & g^{\dagger}_{2,k-1} & g_{n-(k-3)} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ g_{n-1} & g^{\dagger}_{k-1,2} & g^{\dagger}_{k-1,3} & \cdots & g^{\dagger}_{k-1,k-1} & g_n \\ g_n & g^{\dagger}_{k,2} & g^{\dagger}_{k,3} & \cdots & g^{\dagger}_{k,k-1} & g_{n+1} \end{bmatrix},$$

where

$$\begin{aligned} g^{\dagger}_{i,2} &= g_{n-(k-i)} + g_{n-(k-(i-1))}, \\ g^{\dagger}_{i,3} &= g_{n-(k-i)} + g_{n-(k-(i-1))} + g_{n-(k-(i-2))}, \\ &\vdots \\ g^{\dagger}_{i,k-1} &= g_{n-(k-i)} + g_{n-(k-(i-1))} + g_{n-(k-(i-2))} + \cdots + g_{n-(k-(i-(k-2)))}. \end{aligned}$$

Since $Q_k^n Q_k^m = Q_k^{n+m}$, $g_{n+m} = (Q_k^{n+m})_{k,1}$.

Therefore,

$$\begin{aligned} g_{n+m} &= g_n g_{m-(k-1)} + g^{\dagger}_{k,2} g_{m-(k-2)} + g^{\dagger}_{k,3} g_{m-(k-3)} + \cdots \\ &\quad + g^{\dagger}_{k,k-1} g_{m-1} + g_{n+1} g_m \\ &= g_n g_{m-(k-1)} + (g_n + g_{n-1}) g_{m-(k-2)} + (g_n + g_{n-1} + g_{n-2}) g_{m-(k-3)} + \cdots \\ &\quad + (g_n + g_{n-1} + g_{n-2} + \cdots + g_{n-(k-2)}) g_{m-1} + g_{n+1} g_m. \end{aligned}$$

We also have another representation of the nth k-Fibonacci number for positive integers n and m.

Corollary 3.2. For positive integers n and m,

$$\begin{aligned} g_{n+m} &= g_{n-1} g_{m-(k-2)} + (g_{n-1} + g_{n-2}) g_{m-(k-3)} + \\ &\quad (g_{n-1} + g_{n-2} + g_{n-3}) g_{m-(k-4)} + \cdots \\ &\quad + (g_{n-1} + g_{n-2} + g_{n-3} + \cdots + g_{n-(k-1)}) g_m + g_n g_{m+1}. \end{aligned}$$

Proof. Since $g_{n+m} = (Q_k^{n+m})_{k,1} = (Q_k^{n+m})_{k-1,k}$, the proof is completed.

For example, for $n > k$,

$$\begin{aligned} g_{2n} &= g_{2n-1} + g_{2n-2} + \cdots + g_{2n-k} \\ &= g_{n+n} \\ &= g_{n-1} g_{n-(k-2)} + (g_{n-1} + g_{n-2}) g_{n-(k-3)} + \cdots \\ &\quad + (g_{n-1} + g_{n-2} + \cdots + g_{n-(k-1)}) g_n + g_n g_{n+1}. \end{aligned}$$

So, we can get g_{2n} by using $g_{n+1},\ g_n, \ldots,\ g_{n-k+2}$.

The above fact raises a question [1, 2]. What is the relationship between g_n and g_{nt} for a positive integer t. In particular, is there a t such that g_n is a factor of g_{nt}? In the Fibonacci numbers, $F_n | F_{tn}$ for all $t = 1, 2, 3, \ldots$. However, this is not true, in general, for k-Fibonacci numbers, $k \geq 3$.

Lemma 3.3. For any positive integer n, the k-Fibonacci numbers g_{nk+n-k} and $g_{nk+n-k+1}$ are odd numbers.

Proof. If $n = 1$, then $g_1 = g_2 = 1$, i.e., g_1 and g_2 are odd numbers. By induction on n, we may assume true for n, and consider $n+1$.

First,

$$\begin{aligned} g_{(n+1)k+(n+1)-k} &= g_{nk+n+1} \\ &= g_{nk+n} + g_{nk+n-1} + \cdots + g_{nk+n-k+2} + g_{nk+n-k+1} \\ &= g_{nk+n-1} + g_{nk+n-2} + \cdots + g_{nk+n-k+1} + g_{nk+n-k} \\ &\quad + (g_{nk+n-1} + g_{nk+n-2} + \cdots + g_{nk+n-k+2} + g_{nk+n-k+1}) \\ &= 2(g_{nk+n-1} + g_{nk+n-2} + \cdots + g_{nk+n-k+1}) + g_{nk+n-k}. \end{aligned}$$

Then $g_{(n+1)k+(n+1)-k}$ is an odd number since g_{nk+n-k} is an odd number by hypothesis. Similarly, $g_{(n+1)k+(n+1)-k+1}$ is also an odd number.

Therefore, for any positive integer n, the k-Fibonacci numbers g_{nk+n-k} and $g_{nk+n-k+1}$ are odd numbers.

Since $g_n^{(k)} = g_{n-k+2}$, our question can be replaced from "Is there any t such that $g_n^{(k)} | g_{nt}^{(k)}$ for some n?" to "Is there any t such that $g_{n-k+2} | g_{nt-k+2}$ for some n?"

Theorem 3.4. For $k \geq 3$, there exists t such that $g_{n-k+2} \nmid g_{nt-k+2}$ for some n.

Proof. If $k = 3$, then

$$G_3 = (1, 1, 2, 4, 7, 13, 24, 44, 81, 147, \ldots).$$

Here, $g_4 = 4$, $g_9 = 81$ and hence, $g_4 \nmid g_9$. In this case, $n = 5$ and $k = 3$.

Now, suppose that $k \geq 4$. Then, for any positive integer n, the g_{nk+n-k} and $g_{nk+n-k+1}$ are odd numbers. Let $n = k+2$, $t = k$ and let $m = n+1$. Then

$$mt - k + 2 = (n+1)k - k + 2 = \left(n + \frac{n-2}{k}\right) k - k + 2 = nk + n - k.$$

So, g_{mt-k+2} is an odd number. Since $n = k+2$, $k \geq 4$ and $G_k = (1, 1, 2, 4, 8, \ldots)$,

$$g_{m-k+2} = g_{n+1-k+2} = g_5.$$

Since $k \geq 4$, in any cases, $g_5 = 8$. Since g_{m-k+2} is an even number and g_{mk-k+2} is an odd number, there exists t such that $g_{n-k+2} \nmid g_{nt-k+2}$ for some n.

Now we have another question for any positive integers m and n. The question is "how many k-Fibonacci numbers are there between n^m and n^{m+1}?"

Lemma 3.5. For positive integers n and r,

$$ng_r \leq g_{r+n}. \tag{2.3}$$

Proof. If $n = 1$, then $g_r \leq g_{r+1}$. By induction on n, we may assume true for n, and consider $n + 1$. That is,

$$\begin{aligned} ng_r \leq g_{r+n} \;\Rightarrow\; & ng_r + g_r \leq g_{r+n} + g_r \\ \Rightarrow\; & (n+1)g_r \leq g_{r+n-1} + g_{r+n-2} + \cdots + g_{r+n-k} + g_r \\ & = g_{r+n} + (g_{r+n-1} + \cdots + g_{r+n-(k-1)} + g_{r+n-k} + g_r) - g_{r+n} \\ & = g_{r+n+1} + g_{r+n-k} + g_r - g_{r+n}. \end{aligned}$$

Since $g_{r+n} = g_{r+n-1} + g_{r+n-2} + \cdots + g_{r+n-k}$ and $n \geq 1$, $g_{r+n-k} + g_r \leq g_{r+n}$. Thus, $(n+1)g_r \leq g_{r+n+1}$.

Therefore, $ng_r \leq g_{r+n}$ for any positive integers $n,\ r$.

Theorem 3.6. Let m and n be any two positive integers. Then there are no more than n k-Fibonacci numbers between the consecutive powers n^m and n^{m+1}.

Proof. Suppose that the interval between some n^m and n^{m+1} were to contain at least $n+1$ k-Fibonacci numbers:

$$n^m < g_{r+1},\ g_{r+2}, \ldots,\ g_{r+n+1}, \ldots < n^{m+1}.$$

Since $n^m < g_{r+1}$, $n \cdot n^m < ng_{r+1}$. So, by (2.3),

$$n^{m+1} < ng_{r+1} \leq g_{r+n+1}.$$

Consequently, $n^{m+1} < g_{r+n+1}$, a contradiction.

One of the most well-known properties of the Fibonacci sequence is the formula for the sum $S_n^{(2)}$ of the first n terms. A glance at the first few cases quickly leads to the conjecture

$$S_n^{(2)} = F_1 + F_2 + \cdots + F_n = F_{n+2} - 1,$$

which is immediately confirmed by mathematical induction. In case $k \geq 3$, we can easily verify that

$$S_n^{(k)} = \frac{1}{k-1}\left(g_{n+2}^{(k)} - g_{n+(k-2)}^{(k)} - 2g_{n+(k-3)}^{(k)} - \cdots - (k-2)g_{n+1}^{(k)} - 1\right).$$

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