A SPECIAL ABEL'S INTEGRAL EQUATION

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Abstract. In this paper a singular integral equation of Abel's type is solved in a closed form by integration over contours in the complex plane.

1. Introduction. In 1823, Abel, when generalizing the tautochrone problem derived the equation

$$\int_0^x \frac{\varphi(\eta)}{\sqrt{x-\eta}} d\eta = f(x),$$

where $f(x)$ is a given function and $\varphi(\eta)$ is an unknown function. This equation is a particular case of a linear Volterra equation of the first kind.

Abel's problem is the following: Find a curve in the vertical plane $O\xi\eta$ so that a material point, having started its motion at a point of the curve with ordinate x without initial velocity and moving along the curve under the action of gravity without friction, will reach the axis $O\xi$ in time $t = f(x)/\sqrt{2g}$ (g is the acceleration in free falling). If $\beta = \beta(\eta)$ is the angle formed by the tangent to the curve with the axis $O\xi$, then $\varphi(\eta) = 1/\sin\beta$.

Abel's equation is one of the integral equations derived directly from a concrete problem of mechanics or physics (without passing through a differential equation). Historically, Abel's problem is the first one to lead to the study of integral equations.

The generalized Abel's integral equation on a finite segment appeared for the first time in the paper of N. Zeilon [1]. The formalism for reducing the generalized Abel's equation to a characteristic singular equation was expounded there, and it was related to the Hilbert boundary value problem for analytic functions. N. Zeilon concluded that the solution of the generalized Abel's equation was reduced to the consecutive solving of a singular integral equation and the usual Abel's equation, while the solution itself was not given nor was its finding examined. This solution was obtained later by N. I. Muskelishvili [2] and F. D. Gahov [3].

K. D. Sakalyuk [4] gave a solution of the generalized Abel's equation on a finite segment by using the method of T. Carleman for the analytic prolongation [5]. Some constraint assumptions were relinquished by L. von Wolfersdorf [6] and

S. G. Samko [7,8]. In the papers of S. G. Samko [7,8] the solution of the generalized Abel's equation is based on the singular integral operator.

Some modified cases of the generalized Abel's integral equation are given by F. V. Chumakov [9]. M. Orton [10] has given a solution of the generalized Abel's integral equation for a class of generalized functions, and some more general Abel's integral equations on smooth curves were considered by A. S. Peters [11] and H. Neunzert, J. Wich [12]. More details about the generalized Abel's integral equation are given in the monograph [13].

2. Preliminaries. In the present paper we will solve the following special generalized Abel's integral equation

$$\int_a^b \frac{\varphi(t)dt}{(t-x)|t-x|^\alpha} = f(x) \qquad (a < x < b;\ 0 < \alpha < 1). \tag{1}$$

One can verify that the integral equation (1) exists as a singular equation if $\varphi(x)$ belongs to the class of Hölder functions $H(\lambda)$ with exponent λ $(\alpha < \lambda < 1)$.

The integral in this equation can be represented in the form

$$\int_a^b \frac{\varphi(t)dt}{(t-x)|t-x|^\alpha} = \int_a^b \frac{\varphi(t)-\varphi(x)}{(t-x)|t-x|^\alpha}dt + \varphi(x)\int_a^b \frac{dt}{(t-x)|t-x|^\alpha}.$$

The first integral on the right-hand side of the last equality exists as improper if

$\varphi(x) \in H(\lambda)$, and the second one as singular, i.e.

$$\lim_{\varepsilon\to 0}\left[\int_a^{x-\varepsilon}\frac{dt}{(t-x)|t-x|^{\alpha}}+\int_{x+\varepsilon}^{b}\frac{dt}{(t-x)|t-x|^{\alpha}}\right]$$

$$=\lim_{\varepsilon\to 0}\left[-\int_a^{x-\varepsilon}(x-t)^{-\alpha-1}dt+\int_{x+\varepsilon}^{b}(t-x)^{-\alpha-1}dt\right]$$

$$=-\frac{1}{\alpha}\lim_{\varepsilon\to 0}\left[(x-t)^{-\alpha}\Big|_{\alpha}^{x-\varepsilon}+(t-x)^{-\alpha}\Big|_{x+\varepsilon}^{b}\right]$$

$$=-\frac{1}{\alpha}\lim_{\varepsilon\to 0}\left[\varepsilon^{-\alpha}-\frac{1}{(x-a)^{\alpha}}+\frac{1}{(b-x)^{\alpha}}-\varepsilon^{-\alpha}\right]$$

$$=\frac{1}{\alpha(x-a)^{\alpha}}-\frac{1}{\alpha(b-x)^{\alpha}}.$$

Further it follows that

$$\int_a^b\frac{\varphi(t)dt}{(t-x)|t-x|^{\alpha}}=\int_a^b\frac{\varphi(t)-\varphi(x)}{(t-x)|t-x|^{\alpha}}dt+\frac{\varphi(x)}{\alpha}\left[\frac{1}{(x-a)^{\alpha}}-\frac{1}{(b-x)^{\alpha}}\right].$$

We can see from this equality that a class of functions $\varphi(x)$ can be found, for which there exists the integral of the left-hand side in the same sense as $\varphi(x)$ does, up to the endpoints of the segment $[a,b]$, with a characteristic form

$$\varphi(x)=\frac{\varphi^*(x)}{(x-a)^{\gamma_1}(b-x)^{\gamma_2}},$$

where $\varphi^*(x)\in H(\lambda)$ $(0\le\gamma_1\le 1-\alpha,\ 0\le\gamma_2\le 1-\alpha)$.

3. Main Result. First we suppose that $f(x)$ belongs to the class of Hölder's functions $H(\lambda)$ $(0 < \lambda \leq 1)$. If we integrate both sides of the equation (1) over the segment $[a, x]$, we obtain

$$\int_a^b \frac{\varphi(t)dt}{|t-x|^\alpha} - \int_a^b \frac{\varphi(t)dt}{(t-a)^\alpha} = \alpha \int_a^x f(t)dt. \tag{2}$$

Now we represent the function $\varphi(x)$ in the form

$$\varphi(x) = \frac{\psi(x)}{2r(x)} + \frac{\cot \alpha\pi/2}{2\pi r(x)} \cdot \int_a^b \frac{\psi(t)dt}{t-x}, \tag{3}$$

where $\psi(x)$ is an unknown function and

$$r(x) = [(x-a)(b-x)]^{(1-\alpha)/2}.$$

If we substitute (3) in (2), it follows that

$$\frac{1}{2}\int_a^b \frac{\psi(t)dt}{r(t)|t-x|^\alpha} + \frac{\cot \alpha\pi/2}{2\pi} \cdot \int_a^b \frac{1}{r(t)|t-x|^\alpha}\left[\int_a^b \frac{\psi(s)ds}{s-t}\right]dt$$

$$-\frac{1}{2}\int_a^b \frac{\psi(t)dt}{(t-a)^{(1+\alpha)/2}(b-t)^{(1-\alpha)/2}}$$

$$-\frac{\cot \alpha\pi/2}{2\pi} \cdot \int_a^b \frac{1}{(t-a)^{(1+\alpha)/2}(b-t)^{(1-\alpha)/2}}\left[\int_a^b \frac{\psi(s)ds}{s-t}\right]dt = \alpha \int_a^x f(t)dt.$$

In the double integrals of the last equation we change the order of integration and obtain

$$\frac{1}{2}\int_a^b \frac{\psi(t)dt}{r(t)|t-x|^\alpha} - \frac{\cot \alpha\pi/2}{2\pi} \cdot \int_a^b \psi(s)\left[\int_a^b \frac{dt}{r(t)|t-x|^\alpha(t-s)}\right]ds$$

$$-\frac{1}{2}\int_a^b \frac{\psi(t)dt}{(t-a)^{(1+\alpha)/2}(b-t)^{(1-\alpha)/2}} \tag{4}$$

$$+\frac{\cot \alpha\pi/2}{2\pi} \cdot \int_a^b \psi(s)\left[\int_a^b \frac{dt}{(t-a)^{(1+\alpha)/2}(b-t)^{(1-\alpha)/2}(t-s)}\right]ds = \alpha \int_a^x f(t)dt.$$

In the second term of (4), we can decompose the outer integral in two integrals: the first from a to x ($s < x$), and the second from x to b ($s > x$) so that we will calculate the inside integral J_1, once under the assumption that $s < x$ and once assuming that $s > x$. So we will use the complex function

$$F(z) = (z-a)^{(\alpha-1)/2}(z-b)^{(\alpha-1)/2}(z-x)^{-\alpha}(z-s)^{-1} \tag{5}$$

with singularities at $z = a$, $z = b$, $z = x$ and $z = s$, under the assumption $a < s < x$, with suitably chosen branches of the powers of $z - a$, $z - b$ and $z - x$ (see below). The Cauchy theorem can be applied to this function in the region between the contours L_i and L_e (Figure 1):

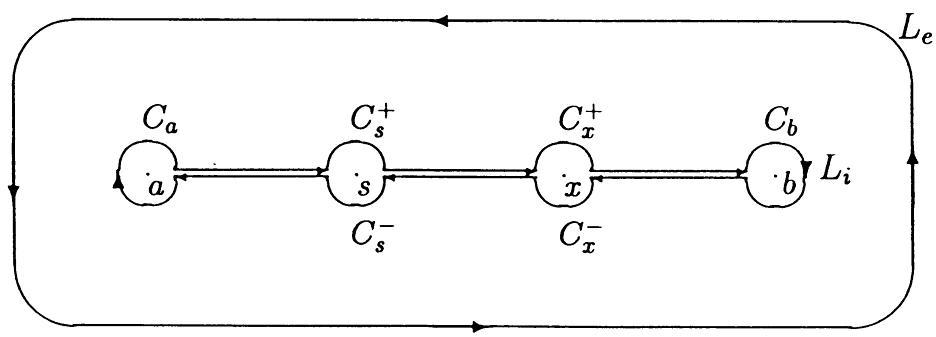

Figure 1

$$\oint_{L_i} F(z)dz = \oint_{L_e} F(z)dz = 2\pi i \ \mathrm{Res}_{z=\infty} F(z). \tag{6}$$

In the Laurent series, if we substitute $z = \infty$ in $1/z^2$, we obtain $\mathrm{Res}_{z=\infty} F(z) = 0$ and the equality (6) takes on the form

$$\oint_{L_i} F(z)dz = 0. \tag{7}$$

Let us denote the radii of the circles C_a, C_b and of the semi-circles C_s^+, C_s^-, C_x^+ and C_x^- by ε. If $\varepsilon \to 0$, at the singular points $z = a$, $z = b$ and $z = x$ of the equality (7), we obtain

$$\lim_{\varepsilon\to 0} \oint_{C_a} F(z)dz = \lim_{\varepsilon\to 0} \oint_{C_b} F(z)dz = \lim_{\varepsilon\to 0} \int_{C_x^\pm} F(z)dz = 0.$$

We further find

$$\lim_{\varepsilon\to 0} \int_{C_s^+} F(z)dz = \lim_{\varepsilon\to 0} \int_\pi^0 F(s+\varepsilon e^{i\theta}) i\varepsilon e^{i\theta} d\theta \tag{8}$$

$$= -\lim_{\varepsilon\to 0} \int_0^\pi \frac{e^{i\pi(\alpha-1)/2} i\varepsilon e^{i\theta} d\theta}{r(s+\varepsilon e^{i\theta})(s+\varepsilon e^{i\theta}-x)^\alpha \varepsilon e^{i\theta}}$$

$$= -ie^{i\pi(\alpha-1)/2} \int_0^\pi \left[\lim_{\varepsilon\to 0} \frac{1}{r(s+\varepsilon e^{i\theta})(s+\varepsilon e^{i\theta}-x)^\alpha}\right] d\theta$$

$$= -\frac{i(-i)e^{-\alpha\pi i/2}}{r(s)(x-s)^\alpha} \int_0^\pi d\theta = -\frac{\pi e^{-\alpha\pi i/2}}{r(s)(x-s)^\alpha}.$$

Analogously we obtain

$$\lim_{\varepsilon\to 0} \int_{C_s^-} F(z)dz = \frac{\pi e^{\alpha\pi i/2}}{r(s)(x-s)^\alpha}. \tag{9}$$

For the value of $F(z)$ on the upper straight-line segments of L_i we have

$$F^+(t) = -\frac{ie^{-\alpha\pi i/2}}{r(t)(x-t)^\alpha(t-s)} \quad (t < x), \tag{10}$$

$$F^+(t) = -\frac{ie^{\alpha\pi i/2}}{r(t)(t-x)^\alpha(t-s)} \quad (t > x),$$

(the arguments of $z-b$ and of $z-x$ for $z=t<x$ are assumed π), and on the lower segments we have

$$F^{-}(t) = -\frac{ie^{\alpha\pi i/2}}{r(t)(x-t)^{\alpha}(t-s)} \quad (t<x), \tag{11}$$

$$F^{-}(t) = -\frac{ie^{-\alpha\pi i/2}}{r(t)(t-x)^{\alpha}(t-s)} \quad (t>x).$$

By substituting (8), (9), (10), and (11) into (7), we obtain

$$\begin{aligned} &- ie^{-\alpha\pi i/2}\int_a^x \frac{dt}{r(t)(x-t)^{\alpha}(t-s)} - ie^{\alpha\pi i/2}\int_x^b \frac{dt}{r(t)(t-x)^{\alpha}(t-s)} \\ &- ie^{\alpha\pi i/2}\int_a^x \frac{dt}{r(t)(x-t)^{\alpha}(t-s)} - ie^{-\alpha\pi i/2}\int_x^b \frac{dt}{r(t)(t-x)^{\alpha}(t-s)} \\ &\qquad = -\frac{\pi(e^{\alpha\pi i/2}-e^{-\alpha\pi i/2})}{r(s)(x-s)^{\alpha}}. \end{aligned}$$

Thus,

$$J_1 = \int_a^b \frac{dt}{r(t)|t-x|^{\alpha}(t-s)} = \frac{\pi\tan\alpha\pi/2}{r(s)(x-s)^{\alpha}} \quad (s<x).$$

Analogously we obtain

$$J_1 = -\frac{\pi\tan\alpha\pi/2}{r(s)(s-x)^{\alpha}} \quad (s>x),$$

which means that

$$J_1 = \operatorname{sgn}(x-s)\frac{\pi\tan\alpha\pi/2}{r(s)|s-x|^{\alpha}}. \tag{12}$$

For the second inside integral J_2 in equation (4) we obtain

$$J_2 = -\frac{\pi \tan \alpha\pi/2}{(s-a)^{(1+\alpha)/2}(b-s)^{(1-\alpha)/2}}. \tag{13}$$

By substituting (12) and (13) in (4), we obtain

$$\int_x^b \frac{\psi(t)dt}{r(t)(t-x)^\alpha} = C + \alpha \int_a^x f(t)dt, \tag{14}$$

where

$$C = \int_a^b \frac{\psi(t)dt}{(t-a)^{(1+\alpha)/2}(b-t)^{(1-\alpha)/2}}.$$

The solution of Abel's integral equation (14) is

$$\psi(x) = r(x)\frac{\sin \alpha\pi}{\pi}\left[\frac{C+\alpha C_1}{(b-x)^{1-\alpha}} - \alpha \int_x^b \frac{f(t)dt}{(t-x)^{1-\alpha}}\right], \tag{15}$$

where

$$C_1 = \int_a^b f(t)dt.$$

By substituting the equality (15) in (3) and after some calculations we obtain

$$\varphi(x) = \frac{\cos \alpha\pi/2}{\pi} \cdot \frac{C+\alpha C_1}{[(x-a)(b-x)]^{(1-\alpha)/2}} - \frac{\alpha \sin \alpha\pi}{2\pi}\int_x^b \frac{f(t)dt}{(t-x)^{1-\alpha}} \tag{16}$$

$$-\frac{\alpha \cos^2 \alpha\pi/2}{\pi^2 r(x)}\int_a^b \frac{r(t)}{t-x}\left[\int_t^b \frac{f(s)ds}{(s-t)^{1-\alpha}}\right]dt.$$

On the other hand, after integration of both sides of (1) over the segment $[x,b]$, we obtain

$$\int_a^b \frac{\varphi(t)dt}{|t-x|^\alpha} - \int_a^b \frac{\varphi(t)dt}{(b-t)^\alpha} = -\alpha \int_x^b f(t)dt.$$

Instead of (3) we will introduce the function

$$\varphi(x) = \frac{\tilde{\psi}(x)}{2r(x)} - \frac{\cot\alpha\pi/2}{2\pi r(x)} \int_a^b \frac{\tilde{\psi}(t)dt}{t-x}.$$

Instead of equation (14), the function $\tilde{\psi}(x)$ satisfies the following Abel's integral equation

$$\int_a^x \frac{\tilde{\psi}(t)dt}{r(t)(x-t)^\alpha} = \tilde{C} - \alpha \int_x^b f(t)dt,$$

where

$$\tilde{C} = \int_a^b \frac{\tilde{\psi}(t)dt}{(t-a)^{(1-\alpha)/2}(b-t)^{(1+\alpha)/2}},$$

and for $\varphi(x)$ we obtain

$$\varphi(x) = \frac{\cos\alpha\pi/2}{\pi} \cdot \frac{\tilde{C}-\alpha C_1}{r(x)} + \frac{\alpha\sin\alpha\pi}{2\pi} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}$$

$$- \frac{\alpha\cos^2\alpha\pi/2}{\pi^2 r(x)} \int_a^b \frac{r(t)}{t-x} \left[\int_a^t \frac{f(s)ds}{(t-s)^{1-\alpha}} \right] dt.$$

By adding together the last equality with (16) we obtain the symmetric expression

$$\varphi(x) = \frac{C}{[(x-a)(b-x)]^{(1-\alpha)/2}} + \frac{\alpha \sin \alpha\pi}{4\pi} \int_a^b \frac{\operatorname{sgn}(x-t)}{|x-t|^{1-\alpha}} f(t)dt \tag{17}$$

$$- \frac{\alpha \cos^2 \alpha\pi/2}{2\pi^2[(x-a)(b-x)]^{(1-\alpha)/2}} \int_a^b \frac{[(t-a)(b-t)]^{(1-\alpha)/2}}{t-x} \left[\int_a^b \frac{f(s)ds}{|s-t|^{1-\alpha}}\right] dt,$$

where C is an arbitrary constant.

Function (17) is a solution of equation (1).

4. Some Particular Cases.

1. If $\alpha = 0$, then equality (17) is in accordance with the results in [2,3].
2. Also, one particular solution for $a = 0$ and $b = x$ is given in [14].
3. If we substitute $a = -\infty$ and $b = +\infty$ in (1), we obtain the integral equation

$$\int_{-\infty}^{\infty} \frac{\varphi(t)dt}{(t-x)|t-x|^\alpha} = f(x) \quad (|x| < \infty,\ 0 < \alpha < 1),$$

and if we apply the Fourier transformation, the solution is given by

$$\varphi(x) = \frac{\alpha}{2\pi} \cot \frac{\alpha}{2\pi} \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-t)}{|x-t|^{1-\alpha}} f(t)dt.$$

This result can be also obtained if we substitute $a = -\infty$ and $b = +\infty$ in (17). This is in accordance with [15,16,17,18].

Remark. There is a possibility for further investigation of the integral equation (1), i.e. its solution can be found for the following classes of functions:

a) bounded at both endpoints a and b ($\gamma_1 = \gamma_2 = 0$);
b) bounded at $x = a$ ($\gamma_1 = 0$) and unbounded at $x = b$ ($0 < \gamma_2 < 1 - \alpha$);
c) bounded at $x = b$ ($\gamma_2 = 0$) and unbounded at $x = a$ ($0 < \gamma_1 < 1 - \alpha$).

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