AN EXTENSION OF DINI'S LEMMA TO NETS OF FUNCTIONS INTO UNIFORM SPACES

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A classical and important lemma of Dini in analysis states:

Theorem D. If $\{f_n\}$ is a sequence of real-valued continuous functions on $[0,1]$ such that $\{f_n(x)\}$ is a non-increasing sequence converging to 0 for each $x \in [0,1]$ then $f_n \to 0$ uniformly.

In this note we extend this lemma to nets of functions mapping into uniform spaces. The following readily established results are used as patterns to formulate and prove the extension.

(1°) A non-increasing sequence $\{x_n\}$ of nonnegative reals converges to 0 if and only if for each $\epsilon > 0$, $\emptyset \neq \{n : x_n - 0 < \epsilon\} \subset \{n : x_k - 0 < \epsilon \text{ for every } k \geq n\}$.

An equivalent form of Theorem D is

(2°) If $\{f_n\}$ is a sequence of real-valued continuous functions on $[0,1]$ such that $\{|f_n(x)|\}$ is a non-increasing sequence converging to 0 for each $x \in [0,1]$ then $f_n \to 0$ uniformly.

These observations motivate the following definitions.

Definition 1. If X is a uniform space with uniformity $\mathcal{U}$ we say a net $\{x_n\}_\Lambda$ converges to $x \in X$ in a monotone fashion if for each $U \in \mathcal{U}$, $\emptyset \neq \{n : (x_n, x) \in U\} \subset \{n : (x_k, x) \in U \text{ for every } k \in \Lambda \text{ with } k \geq n\}$.

Definition 2. Let Y be a uniform space with uniformity $\mathcal{U}$, X be a nonempty set, and $f\colon X \to Y$ a function. We say that a net $\{f_n\}_\Lambda$ of functions from X to Y is pointwise monotonically convergent to f if $\{f_n(x)\}_\Lambda$ converges to $f(x)$ in a monotone fashion for each $x \in X$.

A net $\{f_n(x)\}_\Lambda$ from a topological space X to a uniform space Y with uniformity $\mathcal{U}$ converges to $f\colon X \to Y$ uniformly if for each $U \in \mathcal{U}$, $(f_n(x), f(x)) \in U$ ultimately. For facts on topological and uniform spaces used here without definition see [1].

We are now prepared to present our extension of Dini's lemma.

Theorem. Let X be a compact space and let Y be a uniform space with uniformity $\mathcal{U}$. Let $\{f_n\}_\Lambda$ be a net of continuous functions from X to Y. If $\{f_n\}_\Lambda$ is pointwise monotonically convergent to the continuous function $f\colon X \to Y$ then $\{f_n\}_\Lambda$ converges to f uniformly.

Proof. Suppose $\{f_n\}_\Lambda$ does not converge uniformly to f. Choose a cofinal $\triangle \subset \Lambda$, a net $\{x_n\}_\triangle$ in X and a $U \in \mathcal{U}$ such that each $n \in \triangle$ satisfies $(f_n(x_n), f(x_n)) \notin U$. Choose a symmetric $V \in \mathcal{U}$ such that $V \circ V \circ V = V^3 \subset U$. Since X is compact choose $x \in X$ and a subnet of $\{x_n\}_\triangle$, called again $\{x_n\}_\triangle$, such that $x_n \to x$. Since $f_n(x) \to f(x)$, choose $m \in \triangle$ such that $(f_m(x), f(x)) \in V$. Since f, f_m are continuous at x choose an open set A in X such that $x \in A$, $(f_m(p), f_m(x)) \in V$, $(f(p), f(x)) \in V$, for all $p \in A$. Choose $k \in \triangle$ such that $k > m$ and $x_k \in A$. Then $(f_m(x), f(x)) \in V$, $(f_m(x_k), f_m(x)) \in V$, $(f(x_k), f(x)) \in V$. Therefore $(f_m(x_k), f(x_k)) \in V^3 \subset U$. Since $\{f_n(x_k)\}_\Lambda$ converges to $f(x_k)$ in a monotone fashion and $k > m$ it follows that $f_k(x_k), f(x_k)) \in V^3 \subset U$, a contradiction. The proof is complete.

Reference

1. I. M. James, *Topological and Uniform Spaces*, Springer-Verlag, New York, 1987.

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