NONEMPTY INTERSECTIONS OF MIDDLE α CANTOR SETS

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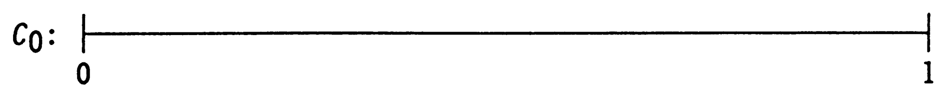
Periodically, one is lucky enough to be able to find an interesting result from current mathematical research that is accessible to undergraduate students. This paper describes such a result from the field of Dynamical Systems. Here we will be exploring the nonempty intersections of two middle α Cantor sets as they are translated across one another. We will present criteria for which the intersection between two such Cantor sets is always nonempty as they are translated across one another. This fact has generated much interest and discussion in some of my upper division classes for mathematics majors and perhaps it can do the same for others.

In the past, most students of mathematics have been introduced to Cantor sets in an introductory course of real analysis. In introductory real analysis, the middle third Cantor set is explored as an example of an uncountable, closed set which contains no interior points or isolated points and has Lebesgue measure zero. More recently, with the popularization of fractals, the middle third Cantor set has become a standard example of a self-similar set. The middle third Cantor set is merely a specific example of a middle α Cantor set, where α has been set to $1/3$.

While the result that we will present here is interesting in its own right, it is also interesting to know some of the motivation for studying intersecting Cantor sets. Earlier we noted that this problem is connected to the discipline of Dynamical Systems. A brief history of why intersections of Cantor sets are important in Dynamical Systems starts in the late 1800's with Poincaré. Poincaré had identified a problem common to understanding many nonlinear dynamical systems, i.e, how to describe changes in the system as a homoclinic bifurcation takes place. It is known that as such a bifurcation takes place, the behavior of a deterministic dynamical system can change dramatically from an easy to understand stable system to a completely chaotic system.

Over the past 20 years, there has been much work done in understanding homoclinic bifurcations (see [9] for a recent overview of the subject). Several major theories have been built around homoclinic bifurcations: omega explosions [8], infinitely many co-existing sinks [1, 7, 10], and antimonotonicity [4]. Each of these theories rely on knowledge of how certain stable and unstable manifolds intersect as homoclinic bifurcations take place. It turns out that these manifolds intersect in the shape of Cantor sets. It is here that the problem of understanding the intersections of Cantor sets as they are translated across one another has become of importance in Dynamical Systems theory.

Recall the standard construction for a middle α Cantor set. First, we let C_0 be the closed interval I of length 1 and $0 < \alpha < 1$. Although we identify I with $[0, 1]$, a middle α Cantor set construction can be carried out in any closed interval of length 1 (or any closed interval in general).



Now, remove from C_0 the open middle interval of length α to obtain C_1 which consists of two closed intervals, each of length β (where $\beta = (1 - \alpha)/2$).

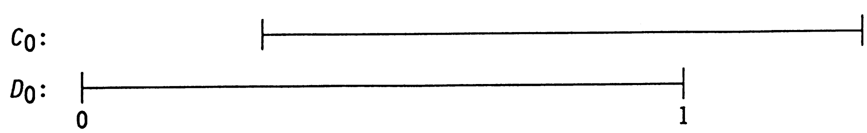
Next, remove the open middle interval (which is of length $\alpha\beta$) from the two closed intervals in C_1 to obtain C_2 which consists of four closed intervals each of length β^2.

This process is continued indefinitely, creating a nested collection of closed nonempty sets $C_0 \supset C_1 \supset C_2 \supset C_3 \supset \cdots$, where each C_n is the union of 2^n intervals, each of length β^n. We define the middle α Cantor set C to be the infinite intersection:

$$C = \bigcap_{n=0}^{\infty} C_n.$$

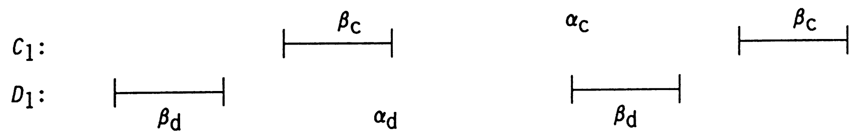
Notice that at every stage in the construction of a middle α Cantor set, each closed interval in C_i is divided into three parts – two closed intervals which move on to the next construction stage and a gap (open interval) which is removed. Let c be a closed interval from C_i. If c has length $l(c)$ prior to being divided, then when c is divided into three parts, the length of each of the two subsequent closed intervals will be $\beta l(c)$ and the length of the subsequent gap will be $\alpha l(c)$. The ratio of the length of the closed intervals to the length of the gap which separates them is always $\frac{\beta l(c)}{\alpha l(c)} = \frac{\beta}{\alpha}$. This common ratio, denoted by $\tau(C)$, will represent the thickness of the middle α Cantor set. The concept of thickness for Cantor sets was first introduced by Newhouse in [6] and was refined in [7].

We will now begin our study of intersecting Cantor sets. Suppose that two (possibly different) middle α Cantor sets C and D are positioned in such a way that they are constructed on top of each other. Then $C \cap D \neq \emptyset$, because they would have at least points 0 and 1 in common. A more challenging situation is the following one. Suppose that the initial construction intervals C_0 and D_0 are arbitrarily positioned such that C_0 overlaps D_0, that is, $C_0 \cap D_0 \neq \emptyset$.



We can think of C_0 as having been translated to the right x units $(0 \leq x \leq 1)$ in relation to D_0. Notice that the intersection of C_0 and D_0 can vary anywhere from the complete interval $[0, 1]$ when $x = 0$, to a single point when $x = 1$.

Now, will $C \cap D \neq \emptyset$? If C is a middle α_c Cantor set, D is a middle α_d Cantor set, and $C \cap D$ is nonempty, then we must have $C_1 \cap D_1 \neq \emptyset$. The only way that $C_1 \cap D_1 = \emptyset$, when $C_0 \cap D_0 \neq \emptyset$, is if $\alpha_d > \beta_c$ and $\alpha_c > \beta_d$,



or $\frac{\beta_c}{\alpha_d} < 1$ and $\frac{\beta_d}{\alpha_c} < 1$. Therefore, $\left(\frac{\beta_c}{\alpha_d}\right)\left(\frac{\beta_d}{\alpha_c}\right) < 1$. If, on the other hand, we have that $\left(\frac{\beta_c}{\alpha_d}\right)\left(\frac{\beta_d}{\alpha_c}\right) \geq 1$, then either $\alpha_d \leq \beta_c$ or $\alpha_c \leq \beta_d$ and $C_1 \cap D_1 \neq \emptyset$. Note that $\left(\frac{\beta_c}{\alpha_d}\right)\left(\frac{\beta_d}{\alpha_c}\right) \geq 1$ is equivalent to $\left(\frac{\beta_c}{\alpha_c}\right)\left(\frac{\beta_d}{\alpha_d}\right) = \tau(C)\tau(D) \geq 1$. Hence, $C_0 \cap D_0 \neq \emptyset$ together with $\tau(C)\tau(D) \geq 1$ implies $C_1 \cap D_1 \neq \emptyset$. Newhouse used the concept of thickness for Cantor sets in [7] to prove a technical result which has had far reaching consequences in the field of Dynamical Systems [4, 6, 8]. It has also spurred further study in the structure of intersecting Cantor sets (e.g., [2, 3, 5, and 11]). The following interesting result, accessible for undergraduates, is a special case of Newhouse's lemma, which is the focus of our study.

Proposition. If C and D are two middle α Cantor sets which satisfy $C_0 \cap D_0 \neq \emptyset$ and $\tau(C)\tau(D) \geq 1$, then $C \cap D \neq \emptyset$.

We will use an inductive proof to show that if $C_0 \cap D_0 \neq \emptyset$ and $\tau(C)\tau(D) \geq 1$, then $C_n \cap D_n \neq \emptyset$ for all $n \geq 0$.

We notice that each of the intersections $C_n \cap D_n$ $(n \geq 0)$ is a closed and bounded set and that $(C_0 \cap D_0) \supset (C_1 \cap D_1) \supset (C_2 \cap D_2) \supset \cdots$. Thus, the Closed Nested Interval Theorem implies the desired result – the following infinite intersection is nonempty:

$$\bigcap_{n=0}^{\infty}(C_n \cap D_n) = C \cap D \neq \emptyset.$$

Earlier we saw that $C_0 \cap D_0 \neq \emptyset$, together with $\tau(C)\tau(D) \geq 1$, forces $C_1 \cap D_1 \neq \emptyset$. This is the base case for our induction argument. Now suppose $C_i \cap D_i \neq \emptyset$; we need to prove that $C_{i+1} \cap D_{i+1} \neq \emptyset$. If $C_i \cap D_i \neq \emptyset$, then at least one of the 2^i closed intervals in C_i must intersect one of the 2^i closed intervals in D_i. Let c and d be a pair of closed intervals ($c \subset C_i$ and $d \subset D_i$) which satisfy the condition $c \cap d \neq \emptyset$. Then the closed intervals c and d will satisfy one of the following two cases.

Case 1. One closed interval is a subset of the other; that is, $c \subset d$ or $d \subset c$. For example:

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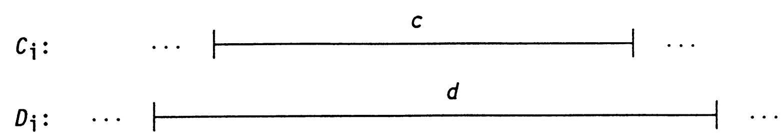
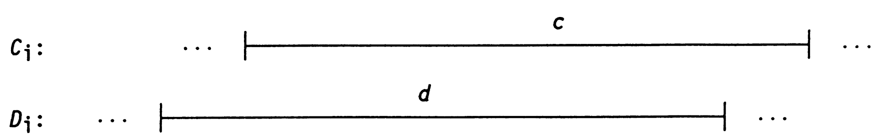
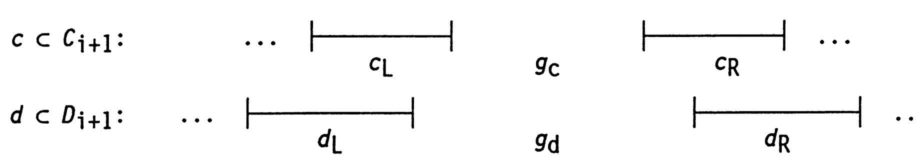


Case 2. Each closed interval is not a subset of the other; $c - (c \cap d) \neq \emptyset$ and $d - (c \cap d) \neq \emptyset$. For example:

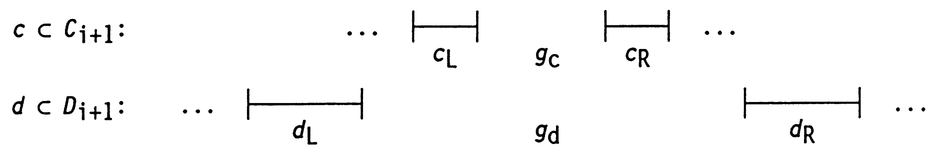


If c is a component of C_i, then in the next or $(i+1)$st stage, c will be divided into two closed intervals denoted by c_L and c_R and a gap denoted by g_c (similarly, d will be divided into d_L, d_R, and g_d):

In the (i)th stage for Case 1, we will assume (without loss of generality) that $c \subset d$. In the (i+1)st stage of the Cantor sets' constructions, we will have one of the two following situations:



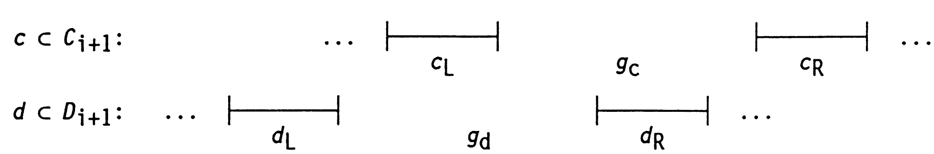
or



That is, either $c \not\subset g_d$ or $c \subset g_d$. If $c \not\subset g_d$, then clearly $C_{i+1} \cap D_{i+1} \neq \emptyset$. In the situation where $c \subset g_d$, we need to recall that there is a bounded gap g adjacent to c (on either the left or right) which satisfies $\frac{l(c)}{l(g)} = \tau(C)$ (where $l(c)$ is the length of c and $l(g)$ is the length of the gap). Suppose that g is on the left of c. Recall that $\frac{l(d_L)}{l(g_d)} = \tau(D)$ and, as we have assumed, $\tau(C)\tau(D) \geq 1$. We now have

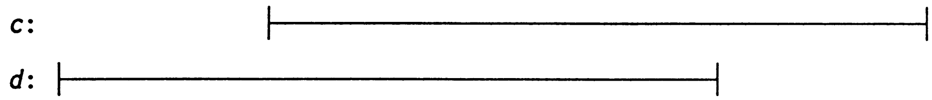
$$1 \leq \tau(C)\tau(D) = \left(\frac{l(c)}{l(g)}\right)\left(\frac{l(d_L)}{l(g_d)}\right) = \left(\frac{l(d_L)}{l(g)}\right)\left(\frac{l(c)}{l(g_d)}\right) < \frac{l(d_L)}{l(g)}.$$

(Since $c \subset g_d$, we have $l(c) < l(g_d)$, which implies $\frac{l(c)}{l(g_d)} < 1$.) Hence, $l(g) \leq l(d_L)$, which in turn implies that d_L intersects the component c' of C_i just to the left of the gap g:

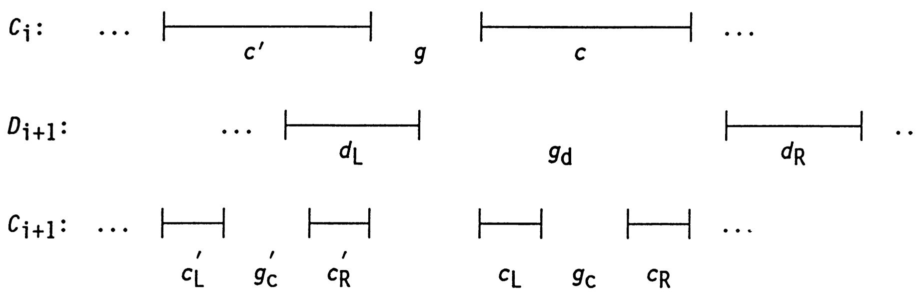


Finally, we see that d_L intersects the right component of c' in C_{i+1}. Therefore, $C_{i+1} \cap D_{i+1} \neq \emptyset$ and the verification of Case 1 is complete.

In the (i)th stage of Case 2, where each closed interval is not a subset of the other, we will assume (without loss of generality) that c and d have the following orientation.



Here, we claim that in the (i+1)st stage, $C_{i+1} \cap D_{i+1} \neq \emptyset$. Assume for a moment that our claim is incorrect. Then we would have the following situation.

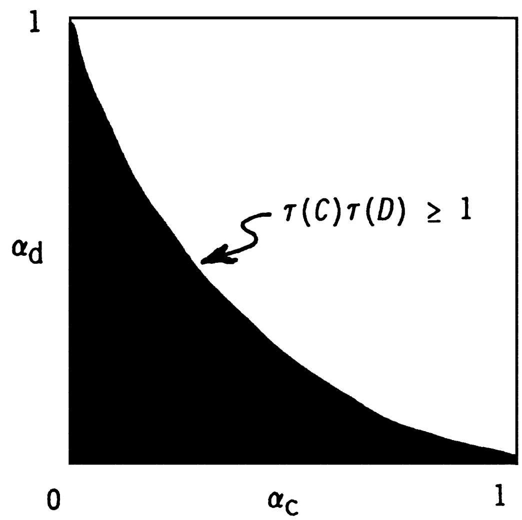


Here, $\frac{l(c_L)}{l(g_d)} < 1$ and $\frac{l(d_R)}{l(g_c)} < 1$. The last two inequalities imply that

$$1 > \left(\frac{l(c_L)}{l(g_d)}\right)\left(\frac{l(d_R)}{l(g_c)}\right) = \left(\frac{l(c_L)}{l(g_c)}\right)\left(\frac{l(d_R)}{l(g_d)}\right) = \tau(C)\tau(D).$$

Since we have assumed $\tau(C)\tau(D) \geq 1$, we have a contradiction. Hence, we must have that $C_{i+1} \cap D_{i+1} \neq \emptyset$. We have now completed our verification of Case 2. This also completes the induction argument and therefore the proof of the proposition is now complete.

Note 1. Cantor sets C and D, which satisfy the hypothesis $\tau(C)\tau(D) \geq 1$, occupy about 38% of the $\alpha_c - \alpha_d$ parameter space. The area of the shaded region in the figure below represents the region where $\tau(C)\tau(D) \geq 1$ and has area $2\ln 2 - 1$.



Note 2. If the hypothesis $\tau(C)\tau(D) \geq 1$ is replaced with $\tau(C)\tau(D) > 1$ and we further assume that $C_0 \cap D_0$ contains more than a single point, then it can be shown that the nonempty intersection $C \cap D$ includes at least one interior point, that is, C and D will intersect at a point which is not an endpoint of the construction intervals.

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