STABLE RANGE IN FORMAL POWER SERIES WITH ANY NUMBER OF INDETERMINATES

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Abstract. Throughout this work (unless otherwise indicated), all rings are commutative rings with identity. Let R be a ring, Λ an index set with cardinality $|\Lambda|$ and let $\{X_\lambda\}_{\lambda\in\Lambda}$ be an arbitrary set of indeterminates over R. In this work for each fixed $i = 1, 2$ or 3 we show the ring $T_i = R[[\{X_\lambda\}_{\lambda\in\Lambda}]]_i$ of formal power series with $|\Lambda|$ indeterminates over R is n-stable (respectively, a B-ring), if and only if R is n-stable (respectively, a B-ring). For each $s \geq 1$, a sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$ of elements in R is said to be stable whenever the ideal $(a_1, a_2, \cdots, a_s, a_{s+1}) = (a_1 + b_1a_{s+1}, \cdots, a_s + b_sa_{s+1})$ for some $b_1, b_2, \cdots, b_s \in R$. A sequence $(a_1, a_2, \cdots, a_s, a_{s+1})$, $a_i \in R$, is said to be unimodular whenever ideal $(a_1, a_2, \cdots, a_s, a_{s+1}) = R$. For any fixed positive integer n we shall say n is in the stable range of R (or simply, R is n-stable), whenever for all $s \geq n$ any unimodular sequence $(a_1, a_2, \cdots, a_s, a_{s+1})$, $a_i \in R$, is stable. R is said to be a B-ring, if for any unimodular sequence $(a_1, a_2, \cdots, a_s, a_{s+1})$, $s \geq 2$, $a_i \in R$ with $(a_1, a_2, \cdots, a_{s-1}) \not\subseteq$ Jacobson radical of R, there exists $b \in R$ such that $(a_1, a_2, \cdots, a_s + ba_{s+1}) = R$.

1. Introduction. In this work (unless otherwise indicated), all rings are commutative rings with identity. Let R be a ring and Λ be an index set with cardinality $|\Lambda|$. Let Z_0 denote the abelian monoid of non-negative rational integers. We assume that the reader is familiar with the concept of the ring $R[X_1, X_2, \cdots, X_n]$ of polynomials with a finite number of indeterminates over R, and also with $R[\{X_\lambda\}_{\lambda\in\Lambda}]$ the ring of polynomials with $|\Lambda|$ indeterminates over R. For references on the ring of polynomials see [3, 6]. By the degree of a monomial $aX_1^{i_1}X_2^{i_2}\cdots X_n^{i_n}(a \in R,\ i_1, i_2, \cdots, i_n \in Z_0)$ we mean the sum of its exponents which is $i_1 + i_2 + \cdots + i_n$. The degree of a nonzero polynomial f which is denoted by ∂f, is the maximum of the degrees of the monomials of which f is the sum. If all the monomials in the sum have the same degree, f is said to be a form. It is clear that if f and g are forms, then fg is either zero or a form of degree $\partial fg = \partial f + \partial g$. A polynomial f of degree m can be expressed uniquely as $f = f_0 + f_1 + \cdots + f_m$, where each f_i is either zero or a form of degree i and f_m cannot be zero.

Now we define the ring of formal power series with a finite number of indeterminates. Let $S = R[X_1, X_2, \cdots, X_n]$ and define S^* to be the set $\{\{f_i\}_0^\infty\}$, where for each $i \in Z_0$, $f_i \in S$ is either zero or a form of degree i. For each $\{f_i\}_0^\infty$ and $\{g_i\}_0^\infty$ in S^*, $\{f_i\} = \{g_i\}$ if and only if $f_i = g_i$ for all $i \in Z_0$, $\{f_i\} + \{g_i\} = \{f_i + g_i\}$ and $\{f_i\}\{g_i\} = \{h_i\}$ where

$$h_i = \sum_{j=0}^{i} f_j g_{i-j}$$

for each $i \in Z_0$. Under the above relation and operations S^* is a ring and is denoted by $R[[X_1, X_2, \cdots, X_n]]$. S^* is called the ring of formal power series with n indeterminates over R. In fact each $\{f_i\}_0^\infty$ can be identified with the formal power series

$$\sum_{i=0}^{\infty} f_i$$

where for each

$$\sum_{i=0}^{\infty} f_i$$

and

$$\sum_{i=0}^{\infty} g_i,$$

addition, multiplication, and equality can be defined as above, correspondingly. It is not difficult to show that S^* has an identity if and only if R has an identity. If $\{X_\lambda\}_{\lambda\in\Lambda}$ is an arbitrary set of indeterminates over R, there are three ways of defining the ring of formal power series for the set $\{X_\lambda\}_{\lambda\in\Lambda}$ over R. We denote these rings by $T_i = R[[\{X_\lambda\}_{\lambda\in\Lambda}]]_i$ where $i = 1, 2$ or 3. We define T_1 to be the set of all rings of the form $R[[X_{\lambda_1}, X_{\lambda_2}, \cdots, X_{\lambda_n}]]$, where $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ runs over all finite subsets of Λ. T_2 is the set of all formal sums

$$\sum_{i=0}^{\infty} f_i$$

where for each i, $f_i \in R[\{X_\lambda\}_{\lambda \in \Lambda}]$ is zero or a form of degree i. Note that equality, addition and multiplication in T_1 and T_2 are defined in the same obvious way as above. For example, let

$$f = \sum f_i$$

and

$$g = \sum g_i$$

be elements in T_1, then $f \in R[[X_{\lambda_1}, X_{\lambda_2}, \cdots, X_{\lambda_n}]]$ and $g \in R[[X_{\lambda'_1}, X_{\lambda'_2}, \cdots, X_{\lambda'_m}]]$ for some subsets $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and $\{\lambda'_1, \lambda'_2, \ldots, \lambda'_m\}$ of Λ. In this case it is obvious that both f and g are in

$$R[[X_{\lambda_1}, X_{\lambda_2}, \cdots, X_{\lambda_n}, X_{\lambda'_1}, \cdots, X_{\lambda'_m}]].$$

Next we show that T_1 (respectively, T_2) has an identity if and only if R has an identity. The necessary condition is immediate since R is a homomorphic image of T_1 (respectively, T_2) under the mapping $\sum f_i \mapsto f_0$. Conversely, it is obvious that if 1 in R is the identity element, then

$$\sum_{i=0}^{\infty} f_i$$

with $f_0 = 1$ and $f_i = 0$ for all $i \geq 1$, is an identity element in T_1 (respectively, T_2).

Let A be an abelian monoid such that for each $a \in A$ there are only a finite number of ordered pairs (b, c) of elements in A with $b + c = a$. Let T be the set of all functions from A into R. For all $a \in A$ and all elements $f, g \in T$ define equality, addition, and multiplication as follows: $f = g$ if and only if $f(a) = g(a)$, $(f+g)(a) = f(a) + g(a)$ and

$$(fg)(a) = \sum_{b+c=a} f(b)g(c).$$

Under this definition T is a ring and whenever A is a direct sum of n copies of Z_0, then T is isomorphic to $R[[X_1, X_2, \cdots, X_n]]$. Now we show that T has an identity if and only if R has an identity. Assume 1_R is the identity element of R, then the function $1_T : A \to R$, given by $1_T(0) = 1_R$ and zero otherwise, is the identity element in T. Conversely, assume f is the identity element in T. In this case we claim $f(0)$ is the identity element in R. Let r be an arbitrary element in R and define $f_r : A \to R$ to be a function that maps zero to r and is zero at other points in A. Now we have

$$rf(0) = f_r(0)f(0) = \sum_{b+c=0} f_r(b)f(c) = (f_r f)(0) = f_r(0) = r.$$

Now we are ready to define T_3. In the above definition of the ring T, if we assume

$$A = \sum (Z_0)_\lambda$$

is the weak direct sum of $|\Lambda|$ copies of Z_0, then we get a ring which is denoted by $T_3 = R[[\{X_\lambda\}_{\lambda\in\Lambda}]]_3$. Next we show that each element $f \in T_3$ can be written as a formal sum

$$\sum_{i=0}^{\infty} f_i$$

where f_i is either zero or a form of degree i. Indeed, f_i in

$$\sum_{i=0}^{\infty} f_i$$

can be a form with infinitely many terms and this is the main difference between T_2 and T_3. Let

$$\{a_\lambda\} \in \sum (Z_0)_\lambda$$

where possibly $a_{\lambda_1} \neq 0, a_{\lambda_2} \neq 0, \cdots, a_{\lambda_n} \neq 0$ for some $\lambda_1, \lambda_2, \cdots, \lambda_n \in \Lambda$. Now for each $r \in R$, define $rX_{\lambda_1}^{a_{\lambda_1}} X_{\lambda_2}^{a_{\lambda_2}} \cdots X_{\lambda_n}^{a_{\lambda_n}}$ to be the function from

$$\sum (Z_0)_\lambda$$

into R such that $(rX_{\lambda_1}^{a_{\lambda_1}} X_{\lambda_2}^{a_{\lambda_2}} \cdots X_{\lambda_n}^{a_{\lambda_n}})(\{a_\lambda\}) = r$ and zero otherwise. Thus, it is clear that each $f \in T_3$ can be expressed as a formal sum of monomials of the form $rX_{\lambda_1}^{a_{\lambda_1}} X_{\lambda_2}^{a_{\lambda_2}} \cdots X_{\lambda_n}^{a_{\lambda_n}}$. From the definition of T_1, T_2 and T_3 it is not difficult to show that T_1 can be embedded into T_2 and T_2 can be embedded into T_3 or simply, we can write $T_1 \subset T_2 \subset T_3$. Actually whenever $|\Lambda|$ is finite, $T_1 = T_2 = T_3$. In other words T_i is independent of the choice of i. For a general reference about T_1, T_2, and T_3, see [2].

In this paper, without having any confusion in the context, we use parentheses to show both the sequence $(a_1, a_2, \cdots, a_s, a_{s+1}), s \geq 1$, of elements in R, and the ideal $(a_1, a_2, \cdots, a_s, a_{s+1})$ generated by $a_1, a_2, \cdots, a_s, a_{s+1} \in R$. A sequence $(a_1, a_2, \cdots, a_s, a_{s+1})$ of elements in R is said to be stable, whenever $(a_1, a_2, \cdots, a_s, a_{s+1}) = (a_1 + b_1 a_{s+1}, \cdots, a_s + b_s a_{s+1})$ for some $b_1, b_2, \cdots, b_s$ in R. A sequence $(a_1, a_2, \cdots, a_s, a_{s+1}), a_i \in R$, is said to be unimodular, if $(a_1, a_2, \cdots, a_s, a_{s+1}) = R$. For a fixed integer $n \geq 1$, we shall say n is in the stable range of R (or simply, R is n-stable), whenever for all $s \geq n$, any unimodular sequence $(a_1, a_2, \cdots, a_s, a_{s+1})$ of elements in R is stable. It is clear, of course, that if R is n-stable, then it is m-stable for any integer $m \geq n$. For more information on stable range in commutative rings, see [1] and [5]. Let $J(R)$ denote the Jacobson radical of R. A ring R is said to be a B-ring whenever for any unimodular sequence $(a_1, a_2, \cdots, a_s, a_{s+1})$, $s \geq 2$, with $(a_1, a_2, \cdots, a_{s-1}) \not\subset J(R)$, there exists $b \in R$ such that $(a_1, a_2, \cdots, a_s + b a_{s+1}) = R$. In fact, we showed in [5] that R is a B-ring if and only if for any unimodular sequence (a_1, a_2, a_3), $a_1, a_2, a_3 \in R$ with $a_1 \notin J(R)$, there exists $b \in R$ such that $(a_1, a_2 + b a_3) = R$. For a detailed study on B-rings, see [4] and [5].

2. Main Results. The following lemma is a result on B-rings which can be found in [4] and a result on n-stable rings which is proved in [5]. Here for the sake of completeness, we state and give a partial proof to this lemma as follows:

Lemma 1. Assume $A \subset J(R)$ is a nonzero proper ideal of R. Then R is n-stable (respectively, a B-ring) if and only if R/A is n-stable (respectively, a B-ring).

Proof. Here we just prove this lemma for n-stable rings. Proof for the case of B-rings which can be found in [4], is left to the reader.

Necessity Part. Let $(a_1 + A, a_2 + A, \cdots, a_s + A, a_{s+1} + A) = R/A$. Hence,

$$1 + A = \sum_{i=1}^{s+1} a_i r_i + A$$

for some $r_1, r_2, \cdots, r_s, r_{s+1} \in R$, implies

$$\left(1 - \sum_{i=1}^{s+1} a_i r_i\right) \in A.$$

Thus, for some $a \in A$ we get $1 \in (a_1, a_2, \cdots, a_s, a_{s+1}r_{s+1} + a)$. Now since R is n-stable, there exists $b_1, b_2, \cdots, b_s \in R$ such that $1 \in (a_1 + b_1(a_{s+1}r_{s+1} + a), \cdots, a_s + b_s(a_{s+1}r_{s+1} + a))$. And now we can conclude that $1 + A \in (a_1 + b_1 r_{s+1} a_{s+1} + A, \cdots, a_s + b_s r_{s+1} a_{s+1} + A)$, which implies R/A is n-stable. Conversely, assume $(a_1, a_2, \cdots, a_s, a_{s+1})$ is a unimodular sequence in R. Thus, we get $1 + A \in (a_1 + A, a_2 + A, \cdots, a_s + A, a_{s+1} + A)$. Since R/A is n-stable, then $1 + A \in (a_1 + b_1 a_{s+1} + A, \cdots, a_s + b_s a_{s+1} + A)$ for some $b_1, b_2, \cdots, b_s \in R$. Thus, for some $a \in A$ and some $X_1, X_2, \cdots, X_s \in R$ we have

$$\sum_{i=1}^{s} (a_i + b_s a_{s+1}) X_i = 1 - a$$

which implies $(a_1 + b_1 a_{s+1}, \cdots, a_s + b_s a_{s+1}) = R$, since $1 - a$ is a unit in R (recall that $a \in A \subset J(R)$).

Remark. It is obvious that the necessity part of the above lemma is still true for any nonzero proper ideal A of R. For more information see [5].

Lemma 2. For each fixed $i = 1, 2$ or 3,

$$f = \left(\sum_{j=0}^{\infty} f_j\right) \in T_i$$

is a unit in T_i if and only if f_0 is a unit in R.

Proof. Since for each $i = 1,\ 2,$ or 3, R is a homomorphic image of T_i under

$$f = \left(\sum_{j=0}^{\infty} f_j\right) \mapsto f_0,$$

thus, the necessity part is clear. In the sufficient part we just give a proof for T_3 and leave the other cases to the reader. Assume f_0 is a unit in R and

$$f = \sum_{j=0}^{\infty} f_j$$

is an element in T_3. In order to show that f is a unit in T_3, it is enough to find an element

$$g = \sum_{j=0}^{\infty} g_j$$

in T_3 such that $fg = 1$. By applying induction we can determine the coefficients of g as follows: $f_0 g_0 = 1$ implies $g_0 = f_0^{-1}$, $f_0 g_1 + f_1 g_0 = 0$ implies $g_1 = -f_0^{-1} f_1 g_0$. Now assume we have $g_0, g_1, g_2, \cdots, g_{k-1}$ and we want to determine g_k. From $f_0 g_k + f_1 g_{k-1} + \cdots + f_k g_0 = 0$ we have $g_k = -f_0^{-1}(f_1 g_{k-1} + f_2 g_{k-2} + \cdots + f_k g_0)$ and notice that here each term in parentheses is either zero or a form of degree k. Thus, g_k is either zero or a form of degree k and the proof by induction is complete.

We showed in [5] that the ring of formal power series $R[[X_1, X_2, \cdots, X_m]]$ with a finite number of indeterminates over R is n-stable (respectively, a B-ring) if and only if R is n-stable (respectively, a B-ring). Next we generalize these results to a formal power series with any number of indeterminates.

Theorem 1. For each fixed $i = 1,\ 2,$ or 3, T_i is n-stable (respectively a B-ring) if and only if R is n-stable (respectively, a B-ring).

Proof. For each $i = 1,\ 2,$ or 3, let $\phi_i : T_i \to R$ be a homomorphism of rings given by

$$f = \left(\sum_{j=0}^{\infty} f_j\right) \mapsto f_0.$$

It is clear that any element

$$f = \sum_{j=0}^{\infty} f_j$$

is in the kernel of ϕ_i ($Ker\phi_i$) if and only if $f_0 = 0$. Thus, by Lemma 2 above, $Ker\phi_i \subset J(T_i)$. Now by Lemma 1 above, the proof of the theorem is complete.

Remark. By using mathematical induction and the fact that $\phi : R[[X]] \to R$ given by $f(X) \mapsto f(0)$ is an epimorphism of rings with $Ker(\phi) \subset J(R[[X]])$, the process of the Proof of Corollaries 2.20 and 2.22 in [5] as mentioned above, is very similar to the argument in the Proof of Theorem 1 above.

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