FROM THE LEGACY OF PYTHAGORAS

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When we first heard about Pythagoras in our early schooldays, we pictured him looking like this (Fig. 1), so impressed were we with the theorem

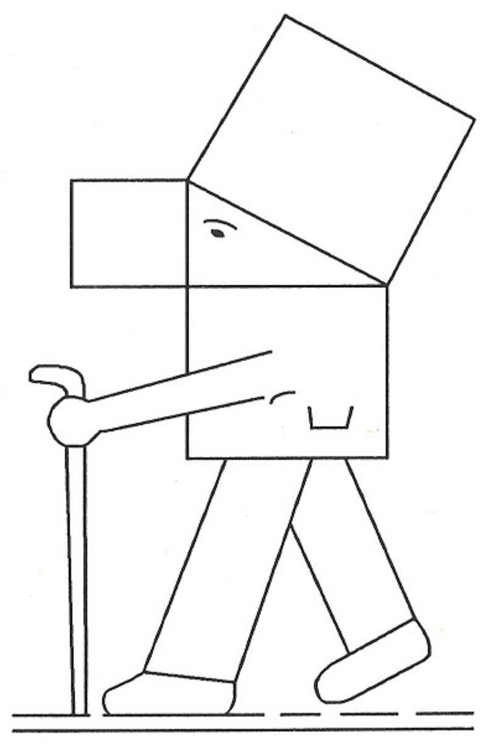


Figure 1.

which bears his name. We did not know then that he (beyond being a magician and a mystic) can also rightly be called *Father of the Theory of Numbers.* Utilizing the advantage of not having paper and writing (or printing) facilities at his disposal, he coined numbers by putting pebbles on the ground. Polygonal (figurate) numbers were born. There were triangular numbers (Fig. 2),

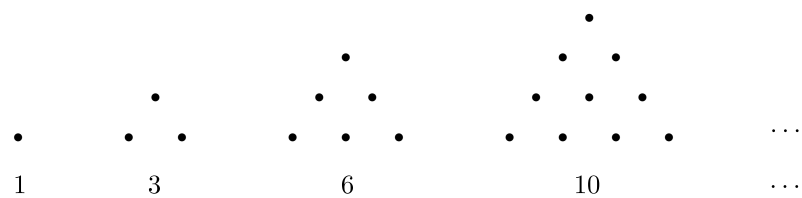


Figure 2.

square numbers (Fig. 3),

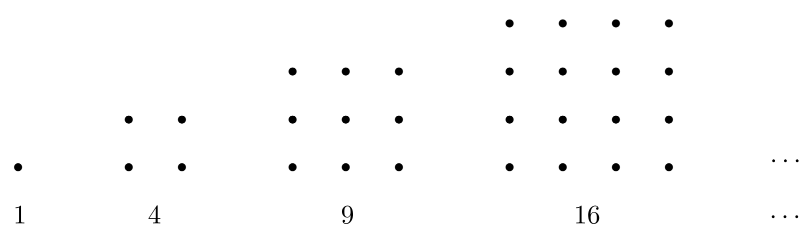


Figure 3.

possibly even pentagonal numbers, and so on. Mathematicians familiar with the fact that

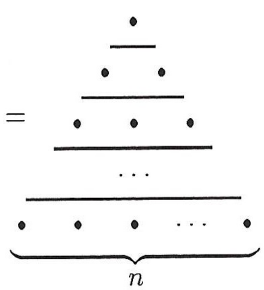
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

and that

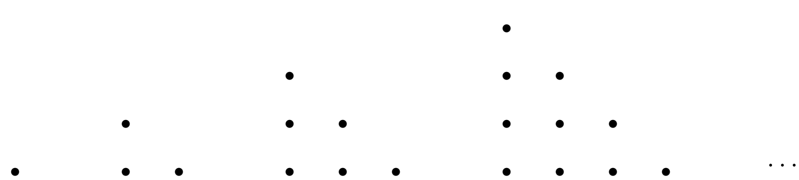
$$\sum_{i=1}^{n} (2i-1) = n^2,$$

rejoice over the fact that these results (and many more) can be conjectured by Pythagoras' ideas of "pebble arithmetic". Thus,

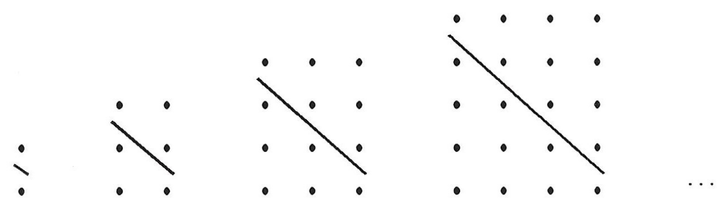
$$\sum_{i=1}^{n} i =$$



which is a triangular number. And, by rearranging triangular numbers



and each time doubling the configuration,



$1 \cdot 2$ $\quad$ $2 \cdot 3$ $\quad$ $3 \cdot 4$ $\quad$ $4 \cdot 5$ $\quad$ $\cdots$

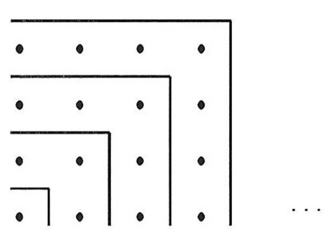
a rectangle of predictable dimensions, namely $n \times (n+1)$ is formed, leading to our well-known formula:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = T_n.$$

For

$$\sum_{i=1}^{n}(2i-1),$$

there is



and

$$\sum_{i=1}^{n}(2i-1)=n^2$$

is foreshadowed. But, mathematicians have been likened to lovers: give them the little finger and they want the whole hand. Letting the n-th polygonal number of k "dimensions" be $P_{n,k}$ ($P_{n,3}$ is the n-th triangular number). We define $P_{n,k}$ recursively.

Definition.

$$P_{n,k}=P_{n-1,k}+[(k-2)n+(3-k)] \quad \text{for} \quad k\geq 3 \quad \text{and all} \quad n.$$

Then

$$P_{n,k}=\frac{n}{2}[(k-2)n+(-k+4)], \tag{1}$$

as can be readily proved by induction on n.

Now we are ready to take the whole hand. Donald P. Skow's problem 83 in this Journal's Spring Edition 1995 (vol. 7, number 2), provided a delightful temptation. With the symbolism introduced, let

$$h=P_{n+2,k}\ P_{n,k}+r\ P_{n+1,k}+m. \tag{2}$$

Is there anything remarkable about h? Using (1) and quite a bit of algebraic manipulation (maybe our students can help)

$$\begin{aligned} h=\frac{1}{4}\{&(k-2)^2n^4+2k(k-2)n^3+(-k^2+12k+2rk-4r-16)n^2\\ &+(-2k^2+8k+2rk)n+(4r+4m)\}. \end{aligned} \tag{3}$$

But, stimulated by Skow, we hope that we can force this expression to become a perfect square. Taking our clue from the first two terms of (3), we stipulate this square to be of the form

$$\left(\frac{(k-2)n^2+kn+a}{2}\right)^2 \tag{4}$$

with the constant $\underline{a}$ yet to be determined. Expanding (4) leads to:

$$\frac{1}{4}\{(k-2)^2n^4+2k(k-2)n^3+(k^2+2ak-4a)n^2+2akn+a^2\}. \tag{5}$$

Comparing the coefficients of equal powers of n in (3) and (5), we have:

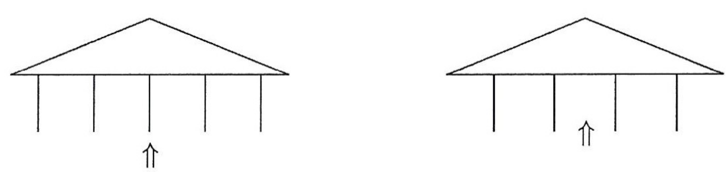
$$k^2+ak-6k-rk-2a+2r+8=0. \tag{6}$$

$$a=-k+r+4. \tag{7}$$

$$4(r+m)=a^2. \tag{8}$$

It follows from (8) that $\underline{a}$ must be even; from (7) that (6) is satisfied; and from (8) that m is integral. Furthermore, k and r must be of the same parity.

(Detour: Pythagoras is said to have been the first to direct our attention to the concept of parity. As the story goes, he looked at temples, pictured below.



and wanting to enter them right in the middle, recognized "bump-in-able" (odd) versus "non-bump-in-able" (even) numbers.)

Furthermore, by (7) and (8):

$$m = \frac{k^2 - 2(r+4)k + (r^2 + 4r + 16)}{4}. \tag{9}$$

Substituting (7) into (4), our square will be

$$\left(\frac{(k-2)n^2 + kn + (r-k+4)}{2}\right)^2,$$

where we yet wish to show that this expression is integral. This means that we want to establish that

$$F(k,n) = n[(k-2)n + k] \equiv 0 \pmod 2.$$

This condition clearly holds if $n \equiv 0 \pmod 2$. If $n \equiv 1 \pmod 2$, $(k-2)n + k$ needs to be even.

Case 1. $k \equiv 0 \pmod 2$. As $(k-2)n \equiv 0 \pmod 2$, $(k-2)n + k \equiv 0 \pmod 2$.

Case 2. $k \equiv 1 \pmod 2$. Here, $(k-2)n \equiv 1 \pmod 2$, hence, $(k-2)n + k \equiv 0 \pmod 2$.

We may now state our Theorem interrelating three consecutive polygonal numbers:

$$P_{n+2,k}P_{n,k} + rP_{n+1,k} + \frac{k^2 - 2(r+4)k + (r^2 + 4r + 16)}{4}$$

$$= \left(\frac{(k-2)n^2 + kn + (r-k+4)}{2}\right)^2.$$

This means that there is an infinitude of relationships among three consecutive polygonal numbers which leads to a square number. Would Pythagoras have liked it? We hope.

References

1. D. P. Skow, Problem 83, *Missouri Journal of Mathematical Sciences*, 7 (1995), 88.

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