ON DENSE METRIZABLE SUBSPACES OF TOPOLOGICAL SPACES

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Abstract. In this note we investigate the question: when does the metrizability of a dense subspace of a topological space imply the metrizability of the whole space? We show that certain conditions always fail to be sufficient and then we examine some elementary examples. We conclude with a theorem which states that a first countable, regular, Hausdorff space Y which has an open metrizable (in the subspace topology) subspace X is metrizable provided $Y - X$ is scattered in Y. Our investigation is conducted on an elementary level.

1. Introduction. A question once asked by a teacher of a topology class (who was a non-specialist) was, "If T is a topological space and X is a dense subset of Y such that X is metrizable in the subspace topology, when can we conclude that Y is metrizable?" I have found no elementary topology textbook that deals with this question. In this note we will investigate this question and then prove a theorem which gives sufficient conditions for Y and $X - Y$ to imply that Y is metrizable.

2. Counterexamples to Some Natural Conjectures. In this section, we will show that certain kinds of metric spaces can always be embedded as a dense subset of a non-metric space Y. Hence, any conjectures about the metrizability must always include certain conditions on X and on Y. We then use certain well-known examples to provide some counterexamples to various "natural" conjectures. Throughout this section, Y is the topological space in question, X is the dense subset of Y which is metrizable in the subspace topology of Y.

Theorem 1. If X is any metric space, X can be embedded as a dense subset of a compact, non-Hausdorff space.

Proof. We use the "open extension" example [1]. Let (X, τ) be a metrizable topological space and p a point not in X. We define a topology τ' for $X \cup \{q\}$ by declaring a set $U \in \tau'$ to be open if $U \in \tau$ or if $U = X \cup \{q\}$. $(X \cup \{q\}, \tau')$ is then a compact non-Hausdorff space with a dense metrizable subspace X.

Theorem 2.. If X is a metric space which has a non-open sequence which has no limit point then X can be embedded as an open dense subspace of a Hausdorff non-regular space.

Proof. Let $\{x_i\}$ be the non-open sequence which has no limit point. Let p be a point not in X and consider $Y = (X \cup \{p\})$. We define a topology for Y as follows: the basic open neighborhoods of all $x \neq \{p\}$ will be the same basic open neighborhoods for x in X and the basic open neighborhoods for p will consist of $p \cup (X - \{x_j\})$ and all

$$W_i^n = p \cup \left(\bigcup_{j=n}^{\infty} B_j\left(\frac{1}{2^i}\right) \right)$$

where $B_j(\epsilon)$ is the ϵ-ball centered at x_j. Note that $\{x_j\}$ remains a closed set in Y. It is easy to see that Y is Hausdorff, given $x_k \in \{x_j\}$ and p note that there exists $\epsilon > 0$ such that $B_k(\epsilon) \cap (\{x_j\} - x_k) = \emptyset$ (by the regularity of X). Choose $n > k$ and i such that $1/2^i < \epsilon/3$. Then $x \in B_k(\epsilon/3)$, $p \in W_i^n$, and $B_k(\epsilon/3) \cap W_i^n = \emptyset$. If $x \notin \{x_j\}$, there exists $\epsilon > 0$ such that $B_x(\epsilon) \cap \{x_j\} = \emptyset$. Choosing i as before, we have $W_i^n \cap B_x(\epsilon/3) = \emptyset$. Hence, Y is Hausdorff. Note that the collection of $W_i^n - \{x_i\}$ is a local basis for p. But any open set containing $\{x_j\}$ must necessarily intersect each $\{W_i^n - \{x_j\}\}$ since each B_i is a local basis for x_i. Hence, there are no open sets separating p from $\{x_j\}$. Hence, Y is not regular.

Theorem 3. If X is a non-second countable metric space then X can be embedded as an open dense subspace of a Hausdorff space that is not first countable.

Proof. Since X is metric and not second countable there exists an uncountable collection of disjoint open sets

$$\bigcup_{\alpha \in I} U_\alpha.$$

(We assume that we have some well ordering on the uncountable index set I.) Let $\{x_\alpha\}$ be a net where $x_\alpha \in U_\alpha$. For each α there exists a ϵ_α (we can assume that for all α that $\epsilon_\alpha < \epsilon$ for some fixed $\epsilon > 0$) such that $x_\alpha \in B_\alpha(\epsilon_\alpha) \subset U_\alpha$. Let $Y = \{X \cup \{p\}\}$ and define the open sets to be the open sets of X together with

$$W_i^\beta = \left(\bigcup_{\alpha > \beta} B_\alpha\left(\frac{\epsilon_\alpha}{i}\right) \right) \cup \{p\}, \quad \text{where } i \in \mathbb{N}.$$

It is clear that Y is not first countable at p. To check that Y is Hausdorff we need only consider p and x where $x \notin \{x_\alpha\}$ and $x \in \{x_\alpha\}$. If $x \in \{x_\alpha\}$, then $x = x_\eta$ and $B_\eta(\epsilon_\eta) \cap W_1^\eta = \emptyset$. If $x \notin \{x_\alpha\}$, then by the regularity of X, there is some $\delta < \epsilon/3$ such that $B_x(\delta) \cap \{x_\alpha\} = \emptyset$. So, $p \in W_3^\beta$ and $W_3^\beta \cap B_x(\delta) = \emptyset$ for some $\beta \in I$.

We will now show, by using some well-known examples, that certain conditions fail to be sufficient.

Example 4. The following is an example of a Hausdorff, regular, first countable, non-metrizable space Y which has a countable dense subspace which is metrizable in the subspace topology.

Let $\mathbb{R}_l$ denote the real line $\mathbb{R}^1$ with the topology generated by basis elements $[x, y)$. Let $\mathbb{Q}_l$ denote the rationals in the subspace topology of $\mathbb{R}_l$. Since $\mathbb{R}_l$ is Hausdorff, regular and first countable, so is $\mathbb{Q}_l$. Since $[q, p)$ $(q, p \in \mathbb{Q})$ forms a basis for the subspace topology of $\mathbb{Q}_l$, $\mathbb{Q}_l$ is second countable and therefore metrizable by the Urysohn metrization theorem. Let q_i be an enumeration of $\mathbb{Q}$. Here is a metric for $\mathbb{Q}_l$:

$$d(x, y) = |x - y| + \sum_{i=1}^{\infty} \left(\frac{1}{2^i}\right) |f_i(x) - f_i(y)|,$$

where $f_i(x) = 1$ if $x \in [q_i, \infty)$ and $f_i(x) = 0$, otherwise. Note that $\mathbb{Q}_l$ with the above metric embeds isometrically into $\mathbb{R}_q$, with the topology of the real line generated by elements of the form $[p, q)$ where p and q are rational. It is a challenging exercise to show that $\mathbb{R}_q$ is homeomorphic to the irrationals in the standard Euclidean topology.

Example 5. An example of a Hausdorff, regular, first countable, separable, non-metrizable space Y which has an open, dense, connected and second countable subset X which is metrizable in the subspace topology and whose complement, $Y - X$, is also metrizable in the subspace topology.

Consider the closed upper half plane $\mathbb{R}^{2+}\{(x, y) \mid y \geq 0\}$ in Niemytzki's tangent disk topology ([1], Example 82). The basic open neighborhoods for points x not on the x-axis are open disks whose boundaries miss the x-axis. The basic open neighborhoods for points on the x-axis together with the tangent point (i.e. $\{(a, 0)\} \cup \{(x, y) \mid x^2 + (y - a)^2 < a^2\}$). Let $X = \{(x, y) \mid y > 0\}$ (the upper open half plane). Note that X is homeomorphic to the open upper half plane in the standard topology via the identity map. $Y - X$ is also metrizable as $Y - X$ in the subspace topology is merely the real line with the discrete topology. But Y is not metrizable as Y is separable but not second countable.

Example 6. An example of a Hausdorff, regular, first countable, locally 1-Euclidean non-metrizable space Y which has an open dense subset X which is metrizable in the subspace topology.

Consider the long line L which is constructed from the ordinal space $[0, \Omega)$ (the set of ordinals $0, 1, 2, \ldots, \omega_0, \omega_0 + 1, \ldots \Omega$ in the standard order topology where ω_0 denotes the least countable ordinal and Ω denotes the least uncountable ordinal) by placing between each ordinal α and its successor $\alpha + 1$ a copy of the standard unit interval $(0, 1)$. See [1], Example 45 or [2], p. 3. The subspace $L - [0, \Omega)$ is an open dense subset of L which is homeomorphic to an uncountable disjoint union of open intervals

$$\bigcup_{\alpha \in [0,\Omega)} (0, 1)_\alpha.$$

$L - [0, \Omega)$ is metrizable in the subspace topology via a standard bounded metric. L is locally 1-Euclidean because every point x of L is contained in an open neighborhood which is homeomorphic to $\mathbb{R}^1$. If $x \in L - [0, \Omega)$ or if x has an immediate predecessor this is clear. If $x \in [0, \Omega)$ and x has no immediate predecessor x is contained in some neighborhood of the form $(\omega, x + 1)$ where ω is some countably infinite ordinal. But (ω, x) is homeomorphic to $\mathbb{R}^1$.

The reader is invited to investigate these concepts and prove theorems of the following type. If X is a dense subset of Y which is metrizable in the subspace topology, then X having properties P and Y having properties Q (and possibly $X - Y$ having properties S) imply that Y is metrizable (or that the metric on X extends to a metric on Y which gives the subspace topology). Here is such a theorem.

Theorem 7. Let Y be a regular, first countable Hausdorff space. Suppose that Y has a dense open subset X which is metrizable in the subspace topology. Furthermore, assume that the set $Y - X$ is a scattered subset of Y (that is, there exist a mutually disjoint collection of open subsets of Y, each of which contains exactly one element of $Y - X$). Then Y is metrizable.

Proof. We assume some well ordering of the elements of $Y - X$ with index set I. Because $Y - X$ is scattered and Y is regular, for each $y_\alpha \in Y - X$, $\alpha \in I$, we get an open set G_α containing y_α where $\overline{G_\alpha} \cap \overline{G_\beta} = \emptyset$ for $\alpha \neq \beta$. By the regularity and first countability of Y, for each y_α we get a countable local basis $\mathcal{U}^\alpha$ such that for each local basis element $U_i^\alpha \in \mathcal{U}^\alpha$ and $i > j$, $\overline{U_i^\alpha} \subset U_j^\alpha$ and $\overline{U_1^\alpha} \subset G_\alpha$.

Because X is metrizable, by the Bing-Nagata-Smirnov Metrization Theorem, X has a basis that is countably locally discrete in X. (Recall that a collection of sets $\mathcal{C}$ is said to be *locally discrete* in Y if every point in Y has an open neighborhood which intersects at most one element of $\mathcal{C}$. $\mathcal{C}$ is said to be *countably locally discrete* if

$$\mathcal{C} = \bigcup_{i=1}^{\infty} \mathcal{C}_i$$

where each $\mathcal{C}_i$ is a locally discrete collection, (see chapter 7 of [3]).

Let

$$\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$$

be a countably locally discrete basis for X. Note that each basis element of X is open in Y. Let

$$\mathcal{B}_n^j = \{B - \bigcup_{\alpha \in I} \overline{U_j^\alpha} \mid B \in \mathcal{B}_n,\ U_j^\alpha \in \mathcal{U}^\alpha\}.$$

Claim.

$$\mathcal{T} = \bigcup_{n=1}^{\infty} \left(\bigcup_{j=1}^{\infty} \mathcal{B}_n^j \cup \left(\bigcup_{\alpha \in I} U_j^\alpha \right) \right)$$

is a countably locally discrete basis for Y. Proof of the claim will prove the Theorem by the Bing-Nagata-Smirnov Metrization Theorem.

Proof of the Claim. It is an easy exercise to see that $\mathcal{T}$ is a countably locally discrete collection of open sets in Y. We need only show that $\mathcal{T}$ is a basis for Y. Let V be an open subset of Y and let $y \in V$. If $y \in Y - X$, we choose the appropriate U_j^α. Suppose now that $y \in X$. If $y \notin G_\alpha$ for any α then $y \notin \cup_{\alpha \in I} \overline{U_1^\alpha}$. Because $\cup_{\alpha \in I} \overline{U_1^\alpha}$ is closed in

Y and Y is regular there is some open set W in Y such that $y \in W \subset Y - \cup_{\alpha \in I} \overline{U_1^\alpha}$. So $y \in W \cap V \subset Y - \cup_{\alpha \in I} \overline{U_1^\alpha} \subset X$. So $W \cap V$ is open in X so there exists an n and a $B \in \mathcal{B}_n$ such that $y \in B \subset W \cap V$. Necessarily $B \in \mathcal{B}_n^1$. Now suppose $y \in G_\alpha$ for some α. Because Y is Hausdorff, we can get an open set W and a U_i^α such that $y \in W$ and $W \cap U_i^\alpha = \emptyset$. Note that $W \cap G_\alpha$ is an open set in X that contains y. We can then find some n and $B \in \mathcal{B}_n$ such that $y \in B \subset W \cap G_\alpha$. Necessarily $B \in \mathcal{B}_n^i$. There $\mathcal{T}$ is a basis for Y.

Examples 5 and 6 show that if $Y - X$ is uncountable, $Y - X$ cannot be allowed to be non-scattered, even when X is assumed to be open in Y. The reader is invited to investigate the case in which $Y - X$ is countable but non-scattered and X is open.

References

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