SOME CHARACTERIZATIONS OF PERFECT NUMBERS

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In this paper, several (perfectly many) characterizations of perfect numbers are presented. A positive integer n is perfect (by definition) if and only if $\sigma(n) = 2n$ where $\sigma(n) = \sum_{d|n} d$ is the sum of all (positive) divisors of n. Replacing d with n/d in the equation $\sum_{d|n} d = 2n$ yields the alternative definition: n is perfect if and only if $\sum_{d|n} 1/d = 2$.

Two very old unsolved problems concerning perfect numbers are:

(i) Do infinitely many perfect numbers exist?

(ii) Do *any* odd perfect numbers exist?

The function σ is an example of a multiplicative function since it has the property $\sigma(mn) = \sigma(m)\sigma(n)$ whenever $\gcd(m,n) = 1$. Other multiplicative functions which appear in the paper are: τ where $\tau(n) = \sum_{d|n} 1$ is the number of divisors of n; Euler's function ϕ where $\phi(n)$ is the number of integers x such that $1 \le x \le n$ and $\gcd(n,x) = 1$; E where $E(n) = n$; U where $U(n) = 1$ for all n; the Moebiüs function μ where $\mu(1) = 1$, $\mu(n) = 0$ if $p^2|n$ for some prime p, $\mu(n) = (-1)^\alpha$ if n is the product of α distinct primes; i where $i(1) = 1$, $i(n) = 0$ for $n > 1$; r where $r(n) = 1/n$; h where $h(n) = n^2$; f where $f(n) = \sigma(n^2)$.

Each of the following 28 formulas is a necessary and sufficient condition for n to be perfect. For the most part, each formula is obtained by substituting $2n$ for $\sigma(n)$ in a more general formula involving σ.

The first three formulas all follow from LaGrange's identity,

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 = \sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 - \sum_{1\le k<j\le n} (a_k b_j - a_j b_k)^2$$

(see [1], page 27). The notation $\sum_{e<d}$ denotes the sum over pairs e and d of divisors of n:

(1) $$\tau(n)\sum_{d|n} d^2 = 4n^2 + \sum_{e<d} (d-e)^2;$$

$$\tau(n) + \sum_{e<d} \frac{e^2 + d^2}{de} = 4n; \tag{2}$$

$$2 \sum_{d|n} d^3 = 4n^2 + \sum_{e<d} \frac{(d^2 - e^2)^2}{de}. \tag{3}$$

To derive (1), using LaGrange's identity (with a_j the j^{th} largest divisor of n and $b_j = 1$),

$$\sigma(n)^2 = \left(\sum_{d|n} d \cdot 1 \right)^2 = \sum_{d|n} d^2 \cdot \sum_{d|n} 1^2 - \sum_{e<d} (d-e)^2,$$

so n is perfect (i.e. $\sigma(n) = 2n$) if and only if

$$4n^2 = \tau(n) \cdot \sum_{d|n} d^2 - \sum_{e<d} (d-e)^2.$$

For formula (2),

$$\tau(n)^2 = \left(\sum_{d|n} 1 \right)^2 = \left(\sum_{d|n} \sqrt{d} \cdot \frac{1}{\sqrt{d}} \right)^2$$

$$= \sum_{d|n} d \cdot \sum_{d|n} \frac{1}{d} - \sum_{e<d} \left(\sqrt{\frac{e}{d}} - \sqrt{\frac{d}{e}} \right)^2$$

$$= \frac{\sigma(n)^2}{n} - \sum_{e<d} \left(\frac{e}{d} + \frac{d}{e} - 2 \right)$$

$$= \frac{\sigma(n)^2}{n} - \sum_{e<d} \frac{e^2 + d^2}{de} + 2 \cdot \sum_{e<d} 1$$

$$= \frac{\sigma(n)^2}{n} - \sum_{e<d} \frac{e^2 + d^2}{de} + \tau(n) \cdot [\tau(n) - 1],$$

and so

$$(a) \qquad \tau(n) + \sum_{e<d} \frac{e^2 + d^2}{de} = \frac{\sigma(n)^2}{n}.$$

Therefore, if n is perfect,

$$\tau(n) + \sum_{e<d} \frac{e^2 + d^2}{de} = 4n.$$

Conversely, if (2) holds, then by (a),

$$\frac{\sigma(n)^2}{n} = 4n,$$

which implies n is perfect.

To obtain formula (3),

$$\sigma(n)^2 = \left(\sum_{d|n} d\right)^2 = \left(\sum_{d|n} \frac{1}{\sqrt{d}} \cdot d\sqrt{d}\right)^2$$

$$= \sum_{d|n} \frac{1}{d} \cdot \sum_{d|n} d^3 - \sum_{e<d} \left(\frac{e\sqrt{e}}{\sqrt{d}} - \frac{d\sqrt{d}}{\sqrt{e}}\right)^2,$$

or

$$(b) \qquad \sigma(n)^2 = \frac{\sigma(n)}{n} \sum_{d|n} d^3 - \sum_{e<d} \frac{(d^2 - e^2)^2}{de}.$$

Replacing $\sigma(n)$ with $2n$ shows that if n is perfect then (3) holds. Conversely, if (3) holds, then using (b) it follows that

$$[\sigma(n) - 2n] \cdot \left(\sum_{d|n} d^3 - n\sigma(n) - 2n^2\right) = 0.$$

It's not difficult to check that

$$\sum_{d|n} d^3 < n\sigma(n) + 2n^2 \quad \text{for } n \leq 3$$

and

$$\sum_{d|n} d^3 > n\sigma(n) + 2n^2 \quad \text{for } n > 3.$$

Therefore, $\sigma(n) - 2n = 0$ and so n is perfect.

Formulas (4)–(17) are based on Abel's partial summation formula (see [1], page 194). In these formulas, d ranges over positive divisors of n, d^+ (for $1 \leq d < n$) is the smallest divisor of n larger than d and d^- (for $(1 < d \leq n)$ is the largest divisor of n smaller than d. For example, if $n = 28$ and $d = 7$, then $d^+ = 14$ and $d^- = 4$. Also, L_d is the sum of the divisors of n which are $\leq d$, i.e.

$$L_d = \sum_{e|n;\ e \leq d} e.$$

Similarly,

$$U_d = \sum_{e|n;\ e \geq d} e;$$

$$F_d = \sum_{e|n;\ e \leq d} \frac{1}{e};$$

$$G_d = \sum_{e|n;\ e \geq d} \frac{1}{e};$$

$$S_d = \sum_{e|n;\ e \leq d} e^2;$$

$$R_d = \sum_{e|n;\ e \geq d} e^2;$$

$$t_d = \sum_{e|n;\ e \leq d} 1 \text{ is the number of factors of } n \text{ which are } \leq d;$$

$$T_d = \sum_{e|n;\ e \geq d} 1.$$

(4) $$\tau(n) = 2 + \sum_{d<n} L_d \left(\frac{1}{d} - \frac{1}{d^+} \right);$$

(5) $$\tau(n) = 2n - \sum_{d>1} U_d \left(\frac{1}{d^-} - \frac{1}{d} \right);$$

(6) $$\tau(n) = 2n - \sum_{d<n} F_d (d^+ - d);$$

(7) $$\tau(n) = 2 + \sum_{d>1} G_d (d - d^-);$$

(8) $$\sum_{d|n} d^2 = 2n^2 - n \sum_{d<n} S_d \left(\frac{1}{d} - \frac{1}{d^+} \right);$$

(9) $$\sum_{d|n} d^2 = 2n + \sum_{d>1} R_d \left(\frac{1}{d^-} - \frac{1}{d} \right);$$

$$2n^2 = 2n + \sum_{d<n} F_d[(d^+)^2 - d^2]; \tag{10}$$

$$2n = 2 + \sum_{d>1} G_d[d^2 - (d^-)^2]; \tag{11}$$

$$\sum_{d|n} d^3 = 2n^3 - \sum_{d<n} L_d[(d^+)^2 - d^2]; \tag{12}$$

$$\sum_{d|n} d^3 = 2n + \sum_{d>1} U_d[d^2 - (d^-)^2]; \tag{13}$$

$$n \cdot [\tau(n) - 2] = \sum_{d<n} t_d(d^+ - d); \tag{14}$$

$$2n = \tau(n) + \sum_{d>1} T_d(d - d^-); \tag{15}$$

$$\sum_{d|n} d^2 = 2n^2 - \sum_{d<n} L_d(d^+ - d); \tag{16}$$

$$\sum_{d|n} d^2 = 2n + \sum_{d>1} U_d(d - d^-). \tag{17}$$

One form of Abel's formula is

$$\sum_{j=1}^{n} a_j b_j = A_n b_n - \sum_{j=1}^{n-1} A_j (b_{j+1} - b_j),$$

where

$$A_j = \sum_{k=1}^{j} a_k.$$

Taking a_j to be the j^{th} largest divisor of n and $b_j = 1/a_j$, it follows that

$$\tau(n) = \sum_{d|n} 1 = \sum_{d|n} d \cdot \frac{1}{d} = \frac{\sigma(n)}{n} - \sum_{d<n} L_d \left(\frac{1}{d^+} - \frac{1}{d} \right).$$

From this, (4) is clear.

Formula (5) is obtained either by repeating the proof as in (4) only using the reverse natural ordering of the divisors of n or else by replacing L_d in (4) with $2n - U_{d^+}$ and simplifying.

For (6), similar to the proof of (4) only taking b_j to be the j^{th} largest divisor of n and $a_j = 1/b_j$,

$$\tau(n) = \sum_{d|n} \frac{1}{d} \cdot d = \frac{\sigma(n)}{n} \cdot n - \sum_{d<n} F_d (d^+ - d).$$

Therefore, n is perfect if and only if (6) holds. Again, formula (7) follows by "reverse ordering" or by putting $F_d = 2 - G_{d^+}$ in (6).

Formulas (8)–(17) are derived in similar fashion:

for (8),

$$\sigma(n) = \sum_{d|n} d^2 \cdot \frac{1}{d} = \frac{1}{n} \sum_{d|n} d^2 - \sum_{d<n} S_d \left(\frac{1}{d^+} - \frac{1}{d} \right);$$

for (10),

$$\sigma(n) = \sum_{d|n} \frac{1}{d} \cdot d^2 = n \cdot \sigma(n) - \sum_{d<n} F_d[(d^+)^2 - d^2];$$

for (12),

$$\sum_{d|n} d^3 = \sum_{d|n} d \cdot d^2 = \sigma(n) \cdot n^2 - \sum_{d<n} L_d[(d^+)^2 - d^2];$$

for (14),

$$\sigma(n) = \sum_{d|n} 1 \cdot d = \tau(n) \cdot n - \sum_{d<n} t_d(d^+ - d);$$

for (16),

$$\sum_{d|n} d^2 = \sum_{d|n} d \cdot d = \sigma(n) \cdot n - \sum_{d<n} L_d(d^+ - d).$$

Formulas (9), (11), (13), (15), (17) are related respectively to (8), (10), (12), (14), (16) in the same way that (5) and (7) are related to (4) and (6).

Formulas (18)–(28) are all based upon Dirichlet multiplication. If f and g are real-valued functions defined on the set $\mathbb{Z}^+$ of positive integers, then the Dirichlet product (or convolution) of f and g is given by

$$(f * g)(n) = \sum_{d|n} f(d) \cdot g(n/d).$$

The following facts are assumed here (see [2], chapter 4): the system of all real-valued multiplicative functions f on $\mathbb{Z}^+$ for which $f(1) \neq 0$ forms an Abelian group under Dirichlet multiplication with identity element i (defined earlier); $\phi * \tau = \sigma$; $\phi * U = E$; $\mu * U = i$. Also, notice that the definitions of σ and τ may be expressed as $\sigma = E * U$ and $\tau = U * U$.

(18) $$(2 - k) \cdot n = \sum_{d|n} \phi(d) \cdot [\tau(n/d) - k], \quad k \text{ any constant};$$

$$\sum_{d<n}[d\cdot\tau(n/d)-\sigma(d)]=n; \tag{19}$$

$$\sum_{1<d<n} d[d\cdot\sigma(n/d)-\sigma(d)]=(n-1)^2; \tag{20}$$

$$n\cdot\tau(n)-3n+1=\sum_{1<d<n}[d\cdot\sigma(n/d)-\tau(d)]; \tag{21}$$

$$\sum_{d<n}\sigma(d)\cdot\phi(n/d)=n[\tau(n)-2]; \tag{22}$$

$$(n+1)\tau(n)-4n=\sum_{1<d<n}[\sigma(d)\cdot\sigma(n/d)-d\cdot\tau(d)\cdot\tau(n/d)]; \tag{23}$$

$$2n(n-1)=\sum_{d<n}\frac{(n^2-d^3)\cdot\sigma(d)}{d^2}; \tag{24}$$

$$\sum_{p|n}\sigma(n/p)-\sum_{p,q|n}\sigma(n/pq)+\sum_{p,q,r|n}\sigma(n/pqr)-+\cdots=n; \tag{25}$$

$$4n=\sum_{d|n}\frac{\sigma(d^2)}{d}; \tag{26}$$

$$2n(2n-1)=\sum_{d<n}[\sigma(n^2/d^2)-\sigma(d)]; \tag{27}$$

$$4n^2 = \sigma(n^2) + n \sum_{d<n} \frac{\tau(\frac{n}{d})\sigma(d^2) - \sigma(d)^2}{d}. \tag{28}$$

To derive (18), if n is perfect then (since $\phi * \tau = \sigma$),

$$\sum_{d|n} \phi(d) \cdot \tau(n/d) = (\phi * \tau)(n) = \sigma(n) = 2n,$$

and

$$\sum_{d|n} k \cdot \phi(d) = k \sum_{d|n} \phi(d) = k \cdot (\phi * U)(n) = k \cdot E(n) = kn.$$

Subtracting these two equations gives (18). Conversely, if (18) holds, then $\sigma(n) - kn = (2-k)n$, so $\sigma(n) = 2n$.

Choosing $k = 0, 1, 2$ in (18) yields the following characterizations of n being perfect: for $k = 0$,

$$\sum_{d|n} \phi(d) \cdot \tau(n/d) = 2n;$$

for $k = 1$,

$$\sum_{d<n} \phi(d)[\tau(n/d) - 1] = n;$$

for $k = 2$,

$$\sum_{\substack{d>1 \\ d \text{ not prime}}} \phi(n/d) \cdot [\tau(d) - 2] = \phi(n).$$

Note that in this last formula, d was replaced by n/d and $\tau(d) = 2$ if and only if d is prime.

For (19), note that $\sigma * U = E * U * U = E * \tau$, and therefore,

$$\sum_{d|n} \sigma(d) = \sum_{d|n} d \cdot \tau(n/d)$$

or

$$\sigma(n) + \sum_{d<n} \sigma(d) = n + \sum_{d<n} d \cdot \tau(n/d).$$

Thus, $\sigma(n) = 2n$ if and only if (19) holds.

The remaining formulas (20)–(28) all have derivations similar to those of (18) and (19). Each results from a corresponding functional identity involving Dirichlet multiplication. The "necessities" are obtained by substituting $2n$ for $\sigma(n)$ in the identity. The "sufficiencies" are straight-forward to check. Note that in (25), $\sum_{p|n}$ denotes the sum over all prime factors of n, $\sum_{p,q|n}$ is the sum over all pairs of distinct prime factors of n, etc.

These functional identities and their derivations are: for (20), $h * \sigma = (E\sigma) * U$ (where $h(n) = n^2$ and $(E\sigma)(n)$ means $E(n) \cdot \sigma(n)$), since

$$(h * E)(n) = \sum_{d|n} d^2 \cdot (n/d) = n \sum_{d|n} d = (E\sigma)(n)$$

and hence, $h * \sigma = h * E * U = (E\sigma) * U$; for (21), $E * \sigma = (E\tau) * U$, since

$$(E * E)(n) = \sum_{d|n} d \cdot (n/d) = n \cdot \sum_{d|n} 1 = (E\tau)(n)$$

and hence, $E * \sigma = E * E * U = (E\tau) * U$; for (22), $\phi * \sigma = E\tau$, since $\phi * \sigma = \phi * U * E = E * E = E\tau$; for (23), $\sigma * \sigma = (E\tau) * \tau$, since $\sigma * \sigma = U * E * U * E = E * E * U * U = (E\tau) * \tau$; for (24), $r * \sigma = (r\sigma) * E$, since

$$(r * U)(n) = \sum_{d|n} \frac{1}{d} = \frac{\sigma(n)}{n} = (r\sigma)(n),$$

and hence, $r * \sigma = r * U * E = (r\sigma) * E$; for (25), $\sigma * \mu = E$, since $E = E * i = E * U * \mu = \sigma * \mu$; for (26), $\sigma^2 = E * f$ (where $f(n) = \sigma(n^2)$), since σ^2 and $E * f$ are both multiplicative and it is straightforward to check that they agree on prime powers; for (27), $\sigma^2 * U = \sigma * f$, since (from (26) above) $\sigma^2 * U = E * f * U = E * U * f = \sigma * f$; for (28), $\sigma^2 * E = (E\tau) * f$ since $\sigma^2 * E = E * f * E = E * E * f = (E\tau) * f$.

References

1. T. Apostol, *Mathematical Analysis*, (second edition), Addison-Wesley, Reading, MA, 1975.

2. I. Niven and H. S. Zuckerman, *An Introduction to the Theory of Numbers*, (fourth edition), John Wiley & Sons, New York, 1980.