SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

65*. [1994, 47] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Evaluate

$$\sum_{k=0}^{n} \left| \binom{n}{k} - 2^k \right|.$$

Comment by the proposer.

I have no solution to this problem. It is equivalent to the question:

$$\text{“When is } 2^k > \binom{n}{k} \text{ ?”}$$

For a related problem, see problem E3327 in the *American Mathematical Monthly*, May 1989, pp. 445–446 and the solution to problem E3327 in the *American Mathematical Monthly*, February 1991, pp. 164–165.

Comment by the editor.

The problem remains open.

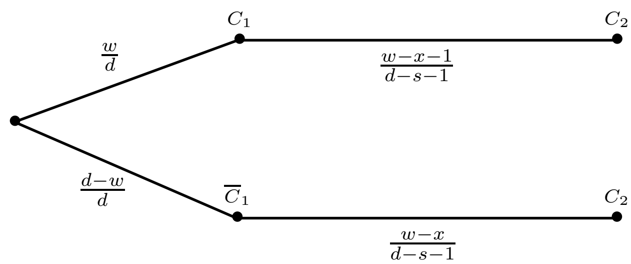
66. [1994, 47] *Proposed by Alvin Beltramo (student), Central Missouri State University, Warrensburg, Missouri.*

Consider the following generalization of the car and the goats problem. A TV host shows you d doors, a car is hidden behind w doors and the rest of the doors are hiding goats. You get to pick a door, winning whatever is behind it. The host, who knows where the cars are, then opens s doors, in the process revealing x cars. The host invites you to switch your choice if you so wish. When should you switch?

Solution by N. J. Kuenzi, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

The probability that you initially pick a door with a car hidden behind it is w/d. A reasonable strategy for switching would be — switch if the probability of picking a door with a car behind it is greater than w/d.

Let C_1 and C_2 denote the events that a car is hidden behind the first and second doors picked, respectively. In order to compute $P(C_2)$, consider the probability tree diagram below:



So

$$P(C_2) = \frac{w}{d}\left(\frac{w-x-1}{d-s-1}\right) + \left(\frac{d-w}{d}\right)\left(\frac{w-x}{d-s-1}\right)$$

$$= \frac{d(w-x)-w}{d(d-s-1)}.$$

When is $P(C_2) > w/d$? Solving yields the following equivalent inequalities:

$$\frac{d(w-x)-w}{d(d-s-1)} > \frac{w}{d},$$

$$d(w-x)-w > w(d-s-1),$$
$$ws > dx,$$

$$\frac{w}{d} > \frac{x}{s}.$$

Hence, $P(C_2) > w/d$ if and only if $x/s < w/d$.

So you should switch and pick a new door if the proportion of cars revealed by the host, x/s, is less than the original proportion of cars, w/d.

Also solved by the proposer.

67. [1994, 47] *Proposed by Leonard L. Palmer, Southeast Missouri State University, Cape Girardeau, Missouri.*

Show that one more than four times the product of two consecutive even or odd numbered triangular numbers is a square.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Larry Hoehn, Austin Peay State University, Clarksville, Tennessee; John Koker, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico; Lawrence Somer, Catholic University of America, Washington, D.C.; Joseph E. Chance, University of Texas-Pan American, Edinburg, Texas; Russell Euler, Northwest Missouri State University, Maryville, Missouri; Donald P. Skow, University of Texas-Pan American, Edinburg, Texas; and the proposer.

$$\begin{aligned}
4T_nT_{n+2}+1 &= 4\left(\frac{n(n+1)}{2}\right)\left(\frac{(n+2)(n+3)}{2}\right)+1 \\
&= [n(n+3)][(n+1)(n+2)]+1 \\
&= (n^2+3n)[(n^2+3n)+2]+1 \\
&= (n^2+3n)^2+2(n^2+3n)+1 \\
&= [(n^2+3n)+1]^2 = (n^2+3n+1)^2.
\end{aligned}$$

Also solved by Herta T. Freitag, Roanoke, Virginia; Melissa Young (student) Hollins College, Roanoke, Virginia; J. Sriskandarajah, University of Wisconsin Center-Richland, Richland Center, Wisconsin; N. J. Kuenzi, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Tony Reese, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; and Hushang Poorkarimi, University of Texas-Pan American, Edinburg, Texas.

Comment by the editor. Can anyone generalize this result?

68. [1994, 48] *Proposed by Joseph B. Dence, University of Missouri-St. Louis, St. Louis, Missouri.*

Let the generalized Fibonacci sequence $\{U_n(P,Q)\}_{n=0}^{\infty}$ be defined by the Binet formula

$$U_n(P,Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $P, Q \in \mathbb{Z}$, $P^2 - 4Q > 0$, and α and β are the unequal roots of $x^2 - Px + Q = 0$. For a particular case denote $U_n(2,-1)$ by P_n, the sequence of Pell numbers. Now consider the series of reciprocals:

$$S = \sum_{k=0}^{\infty} P_{2^k}^{-1}.$$

Show that S is irrational by finding its value explicitly.

Solution I by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

We begin by noting that

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$$

so $P_1 = 1, P_2 = 2$, $P_3 = 5$, $P_4 = 12$, $P_5 = 29$, $P_6 = 70$, $P_7 = 169$, $P_8 = 408, \ldots$.

Consider

$$S(n) = \sum_{j=0}^{n} \frac{1}{P_{2^j}}.$$

$S(1) = 3/2 = 2 - 1/2$, $S(2) = 19/12 = 2 - 5/12$, $S(3) = 647/408 = 2 - 169/408, \ldots$. It appears that

$$(*) \qquad S(n) = 2 - \frac{P_{2^n - 1}}{P_{2^n}}$$

for each positive integer n.

We next establish two preliminary results.

Lemma 1.

$$\frac{P_{2^k-1}}{P_{2^k}} = \frac{(1+\sqrt{2})^{2^{k+1}-1} - (1-\sqrt{2})^{2^{k+1}-1} + 2\sqrt{2}}{(1+\sqrt{2})^{2^{k+1}} - (1-\sqrt{2})^{2^{k+1}}}$$

where k is an arbitrary positive integer.

Proof.

$$\frac{P_{2^k-1}}{P_{2^k}} = \frac{(1+\sqrt{2})^{2^k-1} - (1-\sqrt{2})^{2^k-1}}{(1+\sqrt{2})^{2^k} - (1-\sqrt{2})^{2^k}}$$

$$= \frac{(1+\sqrt{2})^{2^k-1} - (1-\sqrt{2})^{2^k-1}}{(1+\sqrt{2})^{2^k} - (1-\sqrt{2})^{2^k}} \cdot \frac{(1+\sqrt{2})^{2^k} + (1-\sqrt{2})^{2^k}}{(1+\sqrt{2})^{2^k} + (1-\sqrt{2})^{2^k}}$$

$$= \frac{(1+\sqrt{2})^{2^{k+1}-1} - (-1)(1+\sqrt{2}) + (-1)(1-\sqrt{2}) - (1-\sqrt{2})^{2^{k+1}-1}}{(1+\sqrt{2})^{2^{k+1}} - (1-\sqrt{2})^{2^{k+1}}}$$

$$= \frac{(1+\sqrt{2})^{2^{k+1}-1} - (1-\sqrt{2})^{2^{k+1}-1} + 2\sqrt{2}}{(1+\sqrt{2})^{2^{k+1}} - (1-\sqrt{2})^{2^{k+1}}}.$$

Lemma 2.

$$\frac{P_{2^k-1}}{P_{2^k}} - \frac{1}{P_{2^{k+1}}} = \frac{P_{2^{k+1}-1}}{P_{2^{k+1}}}$$

where k is an arbitrary positive integer.

Proof. Using Lemma 1,

$$\frac{P_{2^k-1}}{P_{2^k}} - \frac{1}{P_{2^{k+1}}}$$

$$= \frac{(1+\sqrt{2})^{2^{k+1}-1} - (1-\sqrt{2})^{2^{k+1}-1} + 2\sqrt{2}}{(1+\sqrt{2})^{2^{k+1}} - (1-\sqrt{2})^{2^{k+1}}} - \frac{2\sqrt{2}}{(1+\sqrt{2})^{2^{k+1}} - (1-\sqrt{2})^{2^{k+1}}}$$

$$= \frac{(1+\sqrt{2})^{2^{k+1}-1} - (1-\sqrt{2})^{2^{k+1}-1}}{(1+\sqrt{2})^{2^{k+1}} - (1-\sqrt{2})^{2^{k+1}}} = \frac{P_{2^{k+1}-1}}{P_{2^{k+1}}}.$$

We now proceed to prove $(*)$ using mathematical induction. It is easy to check that the required result holds when $n = 1$. Assume that

$$S(k) = 2 - \frac{P_{2^k-1}}{P_{2^k}}$$

where k is an arbitrary fixed positive integer. Then, using Lemma 2,

$$\begin{aligned} S(k+1) &= \sum_{j=0}^{k+1} \frac{1}{P_{2^j}} = S(k) + \frac{1}{P_{2^{k+1}}} \\ &= \left(2 - \frac{P_{2^k-1}}{P_{2^k}}\right) + \frac{1}{P_{2^{k+1}}} = 2 - \left(\frac{P_{2^k-1}}{P_{2^k}} - \frac{1}{P_{2^{k+1}}}\right) \\ &= 2 - \frac{P_{2^{k+1}-1}}{P_{2^{k+1}}}. \end{aligned}$$

Hence, if the desired result holds when $n = k$, then it also holds when $n = k + 1$. $(*)$ now follows by mathematical induction.

Using the fact that

$$-1 < \frac{1 - \sqrt{2}}{1 + \sqrt{2}} < 0,$$

it follows that

$$S = \sum_{j=0}^{\infty} \frac{1}{P_{2^j}} = \lim_{n\to\infty} \sum_{j=0}^{n} \frac{1}{P_{2^j}}$$

$$= \lim_{n\to\infty} S(n) = \lim_{n\to\infty} \left(2 - \frac{P_{2^n-1}}{P_{2^n}}\right)$$

$$= 2 - \lim_{n\to\infty} \frac{(1+\sqrt{2})^{2^n-1} - (1-\sqrt{2})^{2^n-1}}{(1+\sqrt{2})^{2^n} - (1-\sqrt{2})^{2^n}}$$

$$= 2 - \lim_{n\to\infty} \frac{1 - \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^n-1}}{(1+\sqrt{2}) - \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^n-1}(1-\sqrt{2})}$$

$$= 2 - \frac{1}{1+\sqrt{2}} = 2 - (\sqrt{2}-1) = 3 - \sqrt{2}.$$

Because $S = 3 - \sqrt{2}$, S is irrational.

Solution II by Gerald E. Bergum, South Dakota State University, Brookings, South Dakota.

In [1, page 225], Lucas derived the formula

$$\sum_{n=1}^{\infty} \frac{Q^{2^{n-1}r}}{U_{2^n r}} = \frac{\beta^r}{U_r},\ r \geq 1, \tag{1}$$

where Q and β are as given in the statement of the problem.

Letting $P = 2$, $Q = -1$, and $r = 1$, we have the Pell sequence and (1) becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{2^{n-1}}}{P_{2^n}} = 1 - \sqrt{2} \tag{2}$$

since $U_1 = P_1 = 1$. Expanding the left side of (2) and adding the correct terms to get the proper summation we have

$$S = \sum_{n=0}^{\infty} \frac{1}{P_{2^n}} = 1 - \sqrt{2} + \frac{1}{2} + \frac{1}{2} + 1 = 3 - \sqrt{2}.$$

Hence, S is irrational.

References

1. E. Lucas, "Théorie des fonctions Numériques Simplement Périodiques," *American Journal of Math.*, 1 (1878), 184–240.

Also solved by Wayne McDaniel, University of Missouri-St. Louis, St. Louis, Missouri and the proposer.