AN OPERATOR INEQUALITY

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1. Introduction. In this paper inequalities for positive operators acting on a Hilbert space are considered. For positive operators A and B, the following conjecture [1] was posed:

$$\text{If } 0 \leq B \leq A, \text{ then } (AB^2A)^{1/2} \leq A^2. \tag{1}$$

This conjecture was answered affirmatively by Furuta [3]. Indeed, Furuta proved a more general inequality which contains inequality (1) as a special case:

$$\begin{aligned}&\text{If } 0 \leq B \leq A, \text{ then } (A^r B^p A^r)^{1/q} \leq A^{(p+2r)/q}\\ &\text{for } p, r \geq 0, q \geq 1 \text{ with } p + 2r \leq (1 + 2r)q.\end{aligned}$$

Setting $p = q = 2$ and $r = 1$, Furuta's inequality becomes (1). More interestingly, if one sets $p = 2r$ and $q = 2$, Furuta's inequality becomes a generalization of (1):

$$\text{If } 0 \leq B \leq A, \text{ then } (A^r B^{2r} A^r)^{1/2} \leq A^{2r} \text{ for } r \geq 0. \tag{2}$$

If A and B are positive invertible operators with $B \leq A$, then it is known that $\log B \leq \log A$. In [2], a result of Ando was rephrased: For positive invertible operators A and B, $\log B \leq \log A$ if and only if $(A^r B^{2r} A^r)^{1/2} \leq A^{2r}$ holds for all $r \geq 0$. Thus, Ando's result also establishes (2) as a corollary.

In this paper we establish (2) directly by elementary means. Our approach is inspired by the work of Pedersen and Takesaki [5].

2. Preliminary. We are interested in (bounded, linear) operators acting on a Hilbert space $\mathbb{H}$ with inner product $< .,. >$. An operator A is said to be self-adjoint if $A = A^*$, where A^* is the adjoint of A. A self-adjoint operator A is said to be positive, in notation $A \geq 0$ (or $0 \leq A$), if $< Ax, x > \geq 0$ for every vector $x \in \mathbb{H}$. For positive operators A and B

we write $A \geq B$ (or $B \leq A$) if $A - B \geq 0$. The relation "$\geq$" defines a partial order on the set of positive operators. The following properties of positive operators are well known:

(a) If $A \geq 0$, then $A^r \geq 0$ for every real number $r \geq 0$.
(b) If $A \geq 0$ and A is invertible, then $A^{-1} \geq 0$.
(c) If $A, B \geq 0$, then $ABA \geq 0$. Thus, if $0 \leq B \leq A$, then $CBC \leq CAC$ for every $C \geq 0$.
(d) If $0 \leq B \leq A$, then $B^r \leq A^r$ for $0 \leq r \leq 1$. In particular, if $0 \leq B \leq A$, then $B^{1/2} \leq A^{1/2}$.
(e) In general, $0 \leq B \leq A$ does not necessarily imply $B^2 \leq A^2$.
(f) If $0 \leq T \leq I$, the identity operator, then $T^2 \leq I$.

Properties (c), (d) and (e) led the authors of [1] to pose the conjecture (1). Now we state a weakened version of a result of Pedersen and Takesaki [5] that is suitable for our purpose. For the sake of completeness, a proof is given.

Theorem A. Suppose A and B are positive operators. Also, assume A is invertible. Then, if $(A^{1/2}BA^{1/2})^{1/2} \leq A$, then there is a unique positive operator T satisfying $0 \leq T \leq I$ and $TAT = B$.

Proof. Let $T = A^{-1/2}(A^{1/2}BA^{1/2})A^{-1/2}$. Since $(A^{1/2}BA^{1/2})^{1/2} \leq A$, multiplying both sides of this inequality on the left and on the right by $A^{-1/2}$ yields $0 \leq T \leq I$. A simple calculation shows that $TAT = B$. This establishes the existence of T. To prove the uniqueness, assume S is a positive operator with the property $SAS = B$. Since $SAS = TAT$, multiplying both sides of this equality on the left and on the right by $A^{1/2}$ we obtain $(A^{1/2}SA^{1/2})^2 = (A^{1/2}TA^{1/2})^2$. Now, taking square roots of both sides and then multiplying both sides on the left and on the right by $A^{-1/2}$ produces the desired $S = T$.

3. The Main Result. We are ready to present our proof of (2).

Lemma. If $0 \leq B \leq A$, then $(A^{2^{n-1}}B^{2^n}A^{2^{n-1}})^{1/2} \leq A^{2^n}$ for $n = 0, 1, 2, \ldots$.

Proof. First note that if the inequality can be proven with the extra assumption that the operator A is invertible, then the result follows. For if A is not invertible, then for any $\epsilon > 0$, the operator $A_\epsilon = A + \epsilon I$ is a positive invertible operator. If the inequality in the lemma is derived with A_ϵ in place of A, taking limits as ϵ tends to 0 will produce the desired inequality. Therefore, we may, without loss of generality, assume that A is invertible.

Since $B \leq A$, $(A^{1/2}BA^{1/2})^{1/2} \leq A$. This proves the lemma for $n = 0$. Now, Theorem A implies there is a positive operator $T_1 \leq I$ such that $T_1AT_1 = B$. Thus,

$$AB^2A = A(T_1AT_1)^2A = (AT_1A)T_1^2(AT_1A) \leq (AT_1A)^2.$$

Taking square roots, we have $(AB^2A)^{1/2} \leq AT_1A \leq A^2$. This establishes the lemma for $n = 1$. Again, Theorem A implies there is a positive operator $T_2 \leq I$ such that $T_2A^2T_2 = B^2$. Similar arguments give $(A^2B^4A^2)^{1/2} \leq A^4$. This establishes the lemma for $n = 2$. It is now apparent that the lemma follows by induction.

Notice that for the case $n = 1$, the inequality of the Lemma is (1).

Theorem. If $0 \leq B \leq A$, then $(A^rB^{2r}A^r)^{1/2} \leq A^{2r}$ for $r \geq 0$.

Proof. For each $r \geq 0$, there is a smallest nonnegative integer k such that $r/2^k \leq 1$. Let $A_1 = A^{r/2^k}$ and $B_1 = B^{r/2^k}$. We have $0 \leq B_1 \leq A_1$, $A_1^{2^k} = A^r$ and $B_1^{2^k} = B^r$. The result follows by applying the Lemma with $n = k + 1$ to the operators A_1 and B_1. This completes the proof.

We now use the Theorem to amplify Theorem A when $0 \leq B \leq A$.

Corollary. Suppose $0 \leq B \leq A$ with A invertible. Then, for each $r \geq 0$ there is a unique operator T_r satisfying $0 \leq T_r \leq I$ and $T_rA^rT_r = B^r$.

Proof. Let $T_r = A^{-r}(A^rB^{2r}A^r)^{1/2}A^{-r}$. Clearly we have $T_rA^rT_r = B^r$. Since $B \leq A$, the Theorem implies $0 \leq T_r \leq I$. Arguments similar to those of Theorem A show that the existence of T_r is unique. The proof is complete.

In conclusion, we note that a result of Hansen [4] may be applied to show that the family $\{T_r\}_{r\geq 0}$ in the Corollary is a decreasing family.

References

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