THE IDEAL STRUCTURE OF $\mathbf{Z} * \mathbf{Z}$

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1. Introduction. Let $\mathbb{Z}$ be the set of integers with usual addition and multiplication. Then the Cartesian product $\mathbb{Z} \times \mathbb{Z}$ can be naturally made into a ring via the two operations componentwise addition and multiplication. We will denote this ring by $\mathbb{Z} \times \mathbb{Z}$.

However, there are other operations on the underlying set $\mathbb{Z} \times \mathbb{Z}$ which would make it into a ring. For example, consider the two operations given by,

$$(x, y) + (a, b) = (x + a, y + b)$$
$$(x, y) \cdot (a, b) = (xa, xb + ya + yb)$$

where x, y, a and b are elements of $\mathbb{Z}$. Then it can be shown that the set $\mathbb{Z} \times \mathbb{Z}$ with these operations forms a commutative ring with identity element $(1, 0)$. In this paper, we will denote this new ring by $\mathbb{Z} * \mathbb{Z}$, just to distinguish it from the usual Cartesian product ring $\mathbb{Z} \times \mathbb{Z}$.

The multiplication operation in $\mathbb{Z} * \mathbb{Z}$ seems to be rather unnatural, but it is the same as the multiplication considered in the well known Dorroh Extension Theorem. According to this theorem, any ring R can be embedded in a ring S with identity. To construct S, one would consider the set $\mathbb{Z} \times R$ and define two operations as,

$$(z_1, r_1) + (z_2, r_2) = (z_1 + z_2, r_1 + r_2)$$
$$(z_1, r_1) \cdot (z_2, r_2) = (z_1 z_2, z_1 r_2 + z_2 r_1 + r_1 r_2).$$

It can be shown that the set $\mathbb{Z} \times R$ with the above operations forms a ring with identity element $(1, 0)$. Then denoting this ring by $S = \mathbb{Z} * R$, one can show that the map $f\colon R \to S$ given by $f(r) = (0, r)$ is a ring monomorphism. For more details on Dorroh Extension Theorem the reader can refer to [2] and [4].

In view of this, our ring $\mathbb{Z} * \mathbb{Z}$ can be called "the Dorroh $\mathbb{Z}$ ring". A good question to ask would be, "what is the ideal structure of $\mathbb{Z} * \mathbb{Z}$?" Also of interest is the comparison of the ideal structure of $\mathbb{Z} * \mathbb{Z}$ to that of $\mathbb{Z} \times \mathbb{Z}$. Therefore, it is appropriate to start with some remarks on the old ring $\mathbb{Z} \times \mathbb{Z}$.

It can be shown that all the ideals of $\mathbb{Z} \times \mathbb{Z}$ are of the form $I \times J$, where I and J are ideals of $\mathbb{Z}$. Furthermore, it can be shown that all the prime ideals of $\mathbb{Z} \times \mathbb{Z}$ are of the form $P \times \mathbb{Z}$ or $\mathbb{Z} \times P$ where P is a prime ideal of $\mathbb{Z}$.

Can we expect the same type of result to hold in our new ring $\mathbb{Z} * \mathbb{Z}$? Unfortunately, the answer is in the negative as the following result shows.

Proposition 1.1. Let I be the ideal in $\mathbb{Z} * \mathbb{Z}$ generated by the element $(1,1)$, that is $\mathcal{I} =< (1,1) >$. Then $\mathcal{I}$ cannot be written in the form $I \times J$ for some ideals I and J in $\mathbb{Z}$.

Proof. Notice that $(1,1) \cdot (u,v) = (u, v+u+v) = (u, u+2v)$. Therefore, it follows that $\mathcal{I} = \{(u, u+2v) \mid u, v \in \mathbb{Z}\}$. From this, it can easily be observed that $\mathcal{I} = (\mathcal{E} \times \mathcal{E}) \cup (\mathcal{O} \times \mathcal{O})$ where $\mathcal{E}$ is the set of even integers and $\mathcal{O}$ is the set of odd integers. This will imply that $\mathcal{I}$ cannot be written in the form $I \times J$ for some ideals I and J in $\mathbb{Z}$.

Before investigating the ideal structure of $\mathbb{Z} * \mathbb{Z}$ further, we will record the following two results regarding the invertible elements and zero divisors of $\mathbb{Z} * \mathbb{Z}$.

Proposition 1.2. The only invertible elements of $\mathbb{Z} * \mathbb{Z}$ are $(1,0)$, $(-1,0)$, $(1,-2)$ and $(-1,2)$. In fact,

$$(1,0)^{-1} = (1,0),\ (-1,0)^{-1} = (-1,0),\ (1,-2)^{-1} = (1,-2) \ \text{ and } \ (-1,2)^{-1} = (-1,2).$$

Proof. Suppose (x,y) is an invertible element of $\mathbb{Z} * \mathbb{Z}$. Then $(x,y) \cdot (u,v) = (1,0)$ for some u, v in $\mathbb{Z}$. Hence we obtain that, $(xu, xv + yu + yv) = (1,0)$ which is the same as saying $xu = 1$ and $xv + yu + yv = 0$. It is interesting to notice that adding these last two equations also yields another equation $(x+y)(u+v) = 1$. Hence, our question is equivalent to solving the simultaneous equations $xu = 1$ and $(x+y)(u+v) = 1$ for integer solutions x, y, u and v. The rest is not difficult since the only divisors of 1 are 1 and -1. We will leave the details to the reader.

Remark. One can also show that $\mathbb{Z} * \mathbb{Z}$ has exactly four idempotent elements (see [1], [2], [4] and [6]).

Proposition 1.3. The set S of zero divisors of $\mathbb{Z} * \mathbb{Z}$ is given by

$$S = \{(0,y) \mid y \in \mathbb{Z}\} \cup \{(x,-x) \mid x \in \mathbb{Z}\}.$$

Proof. The proof is a straightforward exercise.

2. Ideals in Z ∗ Z Generated by Two Elements. In this section we will consider the ideals of $\mathbb{Z} * \mathbb{Z}$ generated by two elements. There is an interesting connection between such ideals and integer solutions of certain systems of equations as the following theorem and its corollary show.

Theorem 2.1. Consider the following system of equations with given integer coefficients m_i and n_i, $i = 1, 2$.

$$m_1 x_1 + m_2 x_2 = 1 \tag{1}$$

$$m_1 y_1 + n_1 x_1 + n_1 y_1 + m_2 y_2 + n_2 x_2 + n_2 y_2 = 0. \tag{2}$$

This system of equations has integer solutions for x_i and y_i if and only if $\gcd(m_1, m_2) = 1$ and $\gcd(m_1 + n_1, m_2 + n_2) = 1$.

Proof. The trick is to add the equations (1) and (2). This yields, rather surprisingly, the equation

$$(m_1 + n_1)(x_1 + y_1) + (m_2 + n_2)(x_2 + y_2) = 1.$$

Therefore, our original system of equations is equivalent to the new system

$$\begin{aligned} m_1x_1 + m_2x_2 &= 1 \\ (m_1 + n_1)(x_1 + y_1) + (m_2 + n_2)(x_2 + y_2) &= 1. \end{aligned}$$

Clearly this system has integer solutions for x_i and y_i if and only if the conditions in the theorem are satisfied.

Remark. For given integers m and n simultaneously not equal to zero, $\gcd(m, n)$ denotes the largest positive integer which divides both m and n. If $m = n = 0$, we will use the convention that $\gcd(m, n) = 0$.

Corollary 2.2. Consider the ideal $\mathcal{I}$ in $\mathbb{Z}*\mathbb{Z}$ generated by the two elements (m_1, n_1) and (m_2, n_2), i.e. $\mathcal{I} =< (m_1, n_1), (m_2, n_2) >$. Then $\mathcal{I} = \mathbb{Z} * \mathbb{Z}$ if and only if $\gcd(m_1, m_2) = 1$ and $\gcd(m_1 + n_1, m_2 + n_2) = 1$.

Proof. The proof follows directly from Theorem 2.1, since $\mathcal{I} = \mathbb{Z} * \mathbb{Z}$ if and only if $(1, 0) \in \mathcal{I}$ if and only if there are integers x_1, x_2, y_1 and y_2 such that

$$(m_1, n_1) \cdot (x_1, y_1) + (m_2, n_2) \cdot (x_2, y_2) = (1, 0), \text{ etc.}$$

Remark. The significance of Corollary 2.2 is that it enables us to find out whether a given ideal generated by two elements in $\mathbb{Z} * \mathbb{Z}$ is a proper ideal.

Our next theorem is quite important. It will produce a single generator for an ideal in $\mathbb{Z} * \mathbb{Z}$ generated by two elements.

Theorem 2.3. Consider the ideal $\mathcal{I} =< (m_1, n_1), (m_2, n_2) >$ in $\mathbb{Z} * \mathbb{Z}$. It follows that $\mathcal{I} =< (g_1, g_2 - g_1) >$ where $g_1 = \gcd(m_1, m_2)$ and $g_2 = \gcd(m_1 + n_1, m_2 + n_2)$.

Proof. Let $\mathcal{J} =< (g_1, g_2 - g_1) >$. To show that $\mathcal{J} \subseteq \mathcal{I}$, we will show $(g_1, g_2 - g_1) \in \mathcal{I}$. This is the same as finding integers x_i and y_i such that

$$(g_1, g_2 - g_1) = (m_1, n_1) \cdot (x_1, y_1) + (m_2, n_2) \cdot (x_2, y_2).$$

This reduces to solving the following two equations in $\mathbb{Z}$.

$$m_1x_1 + m_2x_2 = g_1 \tag{3}$$

$$m_1y_1 + n_1x_1 + n_1y_1 + m_2y_2 + n_2x_2 + n_2y_2 = g_2 - g_1. \tag{4}$$

Exactly as in the proof of Theorem 2.1, adding the equations (3) and (4) yields the equation $(m_1+n_1)(x_1+y_1)+(m_2+n_2)(x_2+y_2) = g_2$. Therefore, the question is equivalent to solving the following system in $\mathbb{Z}$.

$$m_1x_1 + m_2x_2 = g_1 \tag{5}$$

$$(m_1 + n_1)(x_1 + y_1) + (m_2 + n_2)(x_2 + y_2) = g_2. \tag{6}$$

Since $\gcd(m_1, m_2) = g_1$, there exist integers x_1 and x_2 such that $m_1x_1 + m_2x_2 = g_1$. Also, since $g_2 = \gcd(m_1 + n_1, m_2 + n_2)$, there exist integers z_1 and z_2 such that

$$(m_1 + n_1)z_1 + (m_2 + n_2)z_2 = g_2.$$

Define $y_i = z_i - x_i$ for $i = 1, 2$. Then it is clear that x_i and y_i satisfy the system of equations (5) and (6). This shows that $\mathcal{J} \subseteq \mathcal{I}$.

Conversely, to prove that $\mathcal{I} \subseteq \mathcal{J}$, one must show that $(m_i, n_i) \in \mathcal{J}$ for $i = 1, 2$. Fix such i. The question is the same as finding integers u_i and v_i such that

$$(m_i, n_i) = (g_1, g_2 - g_1) \cdot (u_i, v_i).$$

This is equivalent to finding integer solutions for u_i and v_i to the following system of equations

$$m_i = g_1u_i \tag{7}$$

$$n_i = g_1v_i + (g_2 - g_1)u_i + (g_2 - g_1)v_i. \tag{8}$$

Add equations (7) and (8) to obtain the new equation $m_i + n_i = g_2(u_i + v_i)$. Hence, the above system is equivalent to the new system

$$m_i = g_1u_i \tag{9}$$

$$m_i + n_i = g_2(u_i + v_i). \tag{10}$$

Since $g_1 = \gcd(m_1, m_2)$, there is an integer u_i such that $m_i = g_1u_i$. On the other hand, since $g_2 = \gcd(m_1 + n_1, m_2 + n_2)$, there is an integer w_i such that $m_i + n_i = g_2w_i$. Define $v_i = w_i - u_i$. Then it is clear that these u_i and v_i satisfy the system of equations given by (9) and (10). This will prove that $\mathcal{I} \subseteq \mathcal{J}$. Hence, the theorem follows.

Remark. The above theorem means that any finitely generated ideal of $\mathbb{Z} * \mathbb{Z}$ is a principal ideal. In other words, $\mathbb{Z} * \mathbb{Z}$ is a Bezout ring (see [3] and [5]). Even though we omit the details here, the same proof can be extended to show that any ideal of $\mathbb{Z} * \mathbb{Z}$ is principal.

The following example will illustrate Corollary 2.2 and Theorem 2.3.

Example. Consider the ideal $\mathcal{I} =< (4, -1), (6, 2) >$ in $\mathbb{Z} * \mathbb{Z}$. Then according to Corollary 2.2, $\mathcal{I}$ must be a proper ideal of $\mathbb{Z} * \mathbb{Z}$ since $\gcd(4, 6) \neq 1$. In addition, since $g_1 = \gcd(4, 6) = 2$ and $g_2 = \gcd(4 + (-1), 6 + 2) = 1$, Theorem 2.3 will imply that $\mathcal{I}$ can be generated by the single element $(2, -1)$.

In the next section, we will investigate the prime and maximal ideals of $\mathbb{Z} * \mathbb{Z}$.

3. Prime and Maximal Ideals of $\mathbf{Z} * \mathbf{Z}$.

Theorem 3.1. All the distinct prime ideals of $\mathbb{Z} * \mathbb{Z}$ are given by

(1) $\mathcal{I}_1 =< (0, 1) >$
(2) $\mathcal{I}_2 =< (1, -1) >$
(3) $\mathcal{I}_3 =< (1, -1 + p) >$
(4) $\mathcal{I}_4 =< (p, 1 - p) >$ where p is any prime.

Proof. Let $\mathcal{I} =< (m, n) >$ be a prime ideal of $\mathbb{Z} * \mathbb{Z}$ where $m, n \in \mathbb{Z}$. Observe that both m and n cannot be simultaneously equal to zero in view of Proposition 1.3. We will first consider the case $m = 0$. Then $n \neq 0$ and $\mathcal{I} = \{(0, nt) \mid t \in \mathbb{Z}\}$. Without loss of generality one can assume that $n > 0$. Let u be any positive divisor of n. Therefore $n = uv$ for some positive integer v. Then it is clear that $(0, n) = (0, u) \cdot (0, v)$. Therefore since $\mathcal{I}$ is a prime ideal, either $(0, u) \in \mathcal{I}$ or $(0, v) \in \mathcal{I}$. Since $\mathcal{I} = \{(0, nt) \mid t \in \mathbb{Z}\}$, it will follow that $n|u$ or $n|v$. However, we know that $u|n$ and $v|n$. Therefore $u = n$ or $v = n$ which implies that $u = n$ or $u = 1$. Therefore $n = 1$ or $n = p$ for some prime p. This means that if $< (0, n) >$ is a prime ideal with $n > 0$, then $n = 1$ or $n = p$ for some prime p. It is not hard to show that $< (0, 1) >$ is a prime ideal of $\mathbb{Z} * \mathbb{Z}$. However, $< (0, p) >$ is not a prime ideal of $\mathbb{Z} * \mathbb{Z}$. This is clear by observing that $(0, p) = (0, 1) \cdot (p - 1, 1)$ but $(0, 1) \notin < (0, p) >$ and $(p - 1, 1) \notin < (0, p) >$.

Next consider the case $m \neq 0$. Without loss of generality, one can assume that $m > 0$. Write $m = xy$ with x and y positive integers. It is easy to observe that there exist $\alpha, \beta \in \mathbb{Z}$ such that $(m, n) = (x, \alpha) \cdot (y, \beta)$. Therefore, since $\mathcal{I}$ is a prime ideal, we will obtain $(x, \alpha) \in \mathcal{I}$ or $(y, \beta) \in \mathcal{I}$. Now suppose that $(x, \alpha) \in \mathcal{I}$. Then

$$(x, \alpha) = (m, n) \cdot (u, v) = (mu, mv + nu + nv) \quad \text{for some} \quad u, v \in \mathbb{Z}.$$

Therefore, $x = mu$ and $m|x$. Similarly, if $(y, \beta) \in \mathcal{I}$, one can obtain that $m|y$. However, since $m = xy$, we know that $x|m$ and $y|m$. Hence, it follows that $x = m$ or $y = m$. This will imply that $m = 1$ or $m = p$ for some prime number p. This means that we have to consider two cases $\mathcal{I} =< (1, n) >$ and $\mathcal{I} =< (p, n) >$ where p is a prime. In either case, the trick is to consider the canonical ring homomorphism $f\colon \mathbb{Z} \to \mathbb{Z} * \mathbb{Z}$ given by $f(z) = (z, 0)$ for $z \in \mathbb{Z}$.

Case I. $\mathcal{I} =< (1, n) >$.

Since $\mathcal{I}$ is a prime ideal of $\mathbb{Z} * \mathbb{Z}$, $f^{-1}(\mathcal{I})$ must be a prime ideal of $\mathbb{Z}$. Therefore, $f^{-1}(\mathcal{I}) = (0)$ or $f^{-1}(\mathcal{I}) = (q)$ for some prime q.

Subcase. $f^{-1}(\mathcal{I}) = (0)$.

In this case we will show that $n = -1$. One can write $\mathcal{I} = \{(u, v + nu + nv) \mid u, v \in \mathbb{Z}\}$. Therefore, whenever u and v are any two integers satisfying $v + nu + nv = 0$, then $u = 0$. Assume that $n + 1 \neq 0$. Define $u = k(n + 1)$ and $v = k - u$ where k is any nonzero integer. Then $u \neq 0$ and one can observe that $v + nu + nv = 0$. This will imply that $u = 0$, which is a contradiction. Therefore $n = -1$. This means, if $\mathcal{I} =< (1, n) >$ is a prime ideal, then $n = -1$. Indeed one can show that $\mathcal{I} =< (1, -1) >$ is a prime ideal of $\mathbb{Z} * \mathbb{Z}$.

Subcase. $f^{-1}(\mathcal{I}) = (q)$ for some prime q.

In this case we will show that $n = q - 1$ or $n = -q - 1$. Since $f(q) = (q, 0) \in \mathcal{I}$, $(q, 0) = (u, v + nu + nv)$ for some $u, v \in \mathbb{Z}$. Therefore, $q = u$ and $0 = v + nu + nv$. Adding these two equations will yield $q = (u + v)(n + 1)$. This will imply the following choices for $u + v$ and $n + 1$.

(a) $u + v = q$ and $n + 1 = 1$.

Therefore, $n = 0$ and $\mathcal{I} =< (1, 0) >$. This will imply that $\mathcal{I} = \mathbb{Z} * \mathbb{Z}$, which is a contradiction.

(b) $u + v = -q$ and $n + 1 = -1$.

Therefore, $n = -2$ and $\mathcal{I} =< (1, -2) >$. One can show that this will also imply that $\mathcal{I} = \mathbb{Z} * \mathbb{Z}$, which is a contradiction.

(c) $u + v = 1$ and $n + 1 = q$.

Hence, $n = q - 1$, and one can, in fact, show that $\mathcal{I} =< (1, q - 1) >$ is a prime ideal of $\mathbb{Z} * \mathbb{Z}$.

(d) $u + v = -1$ and $n + 1 = -q$.

Hence, $n = -q - 1$ and one can show that $\mathcal{I} =< (1, -q - 1) >$ is a prime ideal of $\mathbb{Z} * \mathbb{Z}$. It is not too hard to show that the prime ideal in (c) is equal to the one in (d).

Case II. $\mathcal{I} =< (p, n) >$ where p is a prime.

As in Case I, $f^{-1}(\mathcal{I}) = (0)$ or $f^{-1}(\mathcal{I}) = (q)$ for some prime q.

Subcase. $f^{-1}(\mathcal{I}) = (0)$.

Proceeding as in the first subcase of Case I, one can show that $n = -p$. However, it turns out that $\mathcal{I} =< (p, -p) >$ is not a prime ideal of $\mathbb{Z} * \mathbb{Z}$. We will leave the details to the reader.

Subcase. $f^{-1}(\mathcal{I}) = (q)$ for some prime q.

In this case we will show that $n = 1 - p$ or $n = -1 - p$ as follows. As in Case I, one can write $(q, 0) = (pu, pv + nu + nv)$ for some $u, v \in \mathbb{Z}$. Therefore $q = pu$ and $0 = pv + nu + nv$. The first of these equations will imply that $p|q$, which in turn will imply that $p = q$. Add the two equations to obtain $p = (p + n)(u + v)$. This reduces to the following four cases.

(a) $u + v = p$ and $p + n = 1$.

Therefore, $n = 1 - p$ and one can show that $\mathcal{I} =< (p, 1 - p) >$ is a prime ideal of $\mathbb{Z} * \mathbb{Z}$.

(b) $u+v=-p$ and $p+n=-1$.

Therefore $n=-1-p$ and one can show that $\mathcal{I} =< (p,-1-p) >$ is a prime ideal of $\mathbb{Z}*\mathbb{Z}$. Further it can be shown that this prime ideal is equal to the one in (a).

(c) $u+v=1$ and $p+n=p$.

Therefore $n=0$ and $\mathcal{I} =< (p,0) >$. However, one can show that $< (p,0) >$ is not a prime ideal of $\mathbb{Z}*\mathbb{Z}$. For example, $(p,0)=(1,p-1)\cdot(p,1-p)$ and it is not hard to prove that $(1,p-1)\notin\mathcal{I}$ and $(p,1-p)\notin\mathcal{I}$.

(d) $u+v=-1$ and $p+n=-p$.

Therefore $n=-2p$ and $\mathcal{I} =< (p,-2p) >$. However, one can show that $< (p,-2p) >$ is not a prime ideal of $\mathbb{Z}*\mathbb{Z}$. For example, $(p,-2p)=(1,p-1)\cdot(p,-1-p)$ and it is not difficult to show that $(1,p-1)\notin\mathcal{I}$ and $(p,-1-p)\notin\mathcal{I}$.

The above discussion tells us that the only prime ideals of $\mathbb{Z}*\mathbb{Z}$ are $< (0,1) >$, $< (1,-1) >$, $< (1,-1+p) >$ and $< (p,1-p) >$ where p is a prime. It can also be shown that they are all distinct from each other. Hence the theorem.

Our final theorem describes the maximal ideals of $\mathbb{Z}*\mathbb{Z}$.

Theorem 3.2. All the distinct maximal ideals of $\mathbb{Z}*\mathbb{Z}$ are given by

(1) $\mathcal{I}_3 =< (1,-1+p) >$ and

(2) $\mathcal{I}_4 =< (p,1-p) >$ where p is a prime.

Proof. Since every maximal ideal is a prime ideal, referring to Theorem 3.1, all the maximal ideals must be of the form $\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3$ or $\mathcal{I}_4$. However, $\mathcal{I}_1 =< (0,1) >$ is not a maximal ideal of $\mathbb{Z}*\mathbb{Z}$ since $\mathcal{I}_1\subset\mathcal{I}_4$ for any prime p. This follows by observing that for any p, $(0,1)=(p,1-p)\cdot(0,1)$. Also $\mathcal{I}_2$ cannot be a maximal ideal since $\mathcal{I}_2\subset\mathcal{I}_3$. This is clear because $(1,-1)=(1,-1+p)\cdot(1,-1)$ for any prime p. Therefore, the only candidates for maximal ideals of $\mathbb{Z}*\mathbb{Z}$ are $\mathcal{I}_3$ and $\mathcal{I}_4$.

Let us show $\mathcal{I}_4$ is a maximal ideal of $\mathbb{Z}*\mathbb{Z}$. Consider

$$(\alpha,\beta)\notin\mathcal{I}_4=\{(pu,(1-p)u+v)\mid u,v\in\mathbb{Z}\}.$$

We need to show that $< (p,1-p),(\alpha,\beta) >= \mathbb{Z}*\mathbb{Z}$. But $(\alpha,\beta)\notin\mathcal{I}_4$ means that there are no integers u and v simultaneously satisfying the equations $\alpha=pu$ and $\beta=(1-p)u+v$. However, if $p|\alpha$, it is clear that one can always find $u,v\in\mathbb{Z}$ simultaneously satisfying those two equations. Hence, $p\not|\,\alpha$. Therefore,

$$\begin{aligned}< (p,1-p),(\alpha,\beta) > &=< (\gcd(p,\alpha),\gcd(1,\alpha+\beta)-\gcd(p,\alpha)) >\\ &=< (1,1-1) >=< (1,0) >= \mathbb{Z}*\mathbb{Z}.\end{aligned}$$

This proves that $\mathcal{I}_4$ is a maximal ideal of $\mathbb{Z}*\mathbb{Z}$. In a very similar fashion one can also show that $\mathcal{I}_3$ is a maximal ideal of $\mathbb{Z}*\mathbb{Z}$. Hence, the theorem follows.

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