SIXES AND SEVENS

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Cooper and Kennedy [1] have posed the following interesting question. For $n = 0, 1, \dots,$ let $a_n \in \{6, 7\}$ be such that the base 10 number $a_{n-1}a_{n-2}\dots a_0$ is divisible by 2^n. It is clear that this condition defines the sequence $\{a_n\}$. The first five terms are $6, 7, 7, 7, 6$. Cooper and Kennedy ask if $\{a_n\}$ must contain infinitely many 6's and infinitely many 7's. We show a more general result which immediately answers their question in the affirmative.

Theorem. Let b, c be integers at least 2 and let $\{a_n\}_{n=0}^{\infty}$ be a sequence with each $a_n \in \{0, 1, \dots, b-1\}$. Suppose that for each positive integer m there is some integer N_m, such that c^m divides $a_0 + a_1 b + \cdots + a_n b^n$ for each $n \ge N_m$. If $\{a_n\}$ is eventually periodic, then $\{a_n\}$ is identically zero.

Proof. It clearly suffices to prove the result in the case that $c = p$, a prime. First note that the hypothesis that N_m always exists implies that either $\{a_n\}$ is identically zero or $p|b$. Indeed, let

$$A_n := \sum_{i=0}^{n-1} a_i b^i$$

for $n = 1, 2, \dots$. Let m be so large that $p^m \ge b$. If $n > N_m$, then

$$p^m | A_n, \quad p^m | A_{n+1} = A_n + a_n b^n,$$

so that $p^m | a_n b^n$. But $0 \le a_n \le b - 1$ and $p^m \ge b$. Thus, either $\{a_n\}$ is eventually zero or $p|b$. But if $\{a_n\}$ is eventually zero, then there is some N with $A_n = A_N$ for all $n \ge N$. Thus, $p^m | A_N$ for all m, so that $A_N = 0$ and $\{a_n\}$ is identically zero.

Thus, we may assume $p|b$. Suppose $\{a_n\}$ is eventually periodic, so that there are integers N, k with $a_n = a_{n+k}$ for all $n \geq N$. Let

$$B := \sum_{i=0}^{k-1} a_{N+i} b^i.$$

Thus, for $j = 1, 2, \ldots$, we have

$$A_{N+jk} = A_N + Bb^N(1 + b^k + \cdots + b^{(j-1)k}) = A_N + Bb^N \frac{b^{jk} - 1}{b^k - 1},$$

so that

$$A_N(b^k - 1) - Bb^N = A_{N+jk}(b^k - 1) - Bb^{N+jk}. \tag{1}$$

By our hypothesis and by $p|b$, the right side of (1) is divisible by arbitrarily high powers of p as $j \to \infty$. But the left side of (1) is constant, so it must be 0. Thus,

$$A_N(b^k - 1) = Bb^N. \tag{2}$$

We conclude that $b^N | A_N$. But $0 \leq A_N < b^N$, so that $A_N = 0$. Hence (2) implies that $B = 0$, from which it follows that $\{a_n\}$ is identically zero. This completes the proof of the theorem.

Reference

1. C. Cooper and R. E. Kennedy, Private communication, 14 January, 1994.