SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

53. [1993, 39] *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Prove analytically that

$$\sqrt[3]{19+9\sqrt{6}}+\sqrt[3]{19-9\sqrt{6}}$$

is an integer.

Solution I by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Gregory Bruton, Cape Girardeau, Missouri; and Sherri Palmer, Ste. Genevieve, Missouri.

Since $(1+\sqrt{6})^3 = 19+9\sqrt{6}$ and $(1-\sqrt{6})^3 = 19-9\sqrt{6}$, the desired sum equals 2.

Solution II by Robert L. Doucette, McNeese State University, Lake Charles, Louisiana; Donald P. Skow, University of Texas-Pan American, Edinburg, Texas; Kanadasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Joseph E. Chance, University of Texas-Pan American, Edinburg, Texas; Seung-Jin Bang, Albany, California; Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico; J. Sriskandarajah, University of Wisconsin Center-Richland, Richland Center, Wisconsin; and the proposer.

Let $s = \sqrt[3]{19+9\sqrt{6}}$ and $t = \sqrt[3]{19-9\sqrt{6}}$. Note that $st = \sqrt[3]{19^2-9^2 6} = -5$ and $s^3+t^3 = (19+9\sqrt{6})+(19-9\sqrt{6}) = 38$. Since

$$(s+t)^3 = s^3+3s^2t+3st^2+t^3 = (s^3+t^3)+3st(s+t) = 38-15(s+t),$$

we see that $s+t$ is a root of the equation $x^3+15x-38=0$. Noting that $x=2$ is a root of this equation, we have $x^3+15x-38 = (x-2)(x^2+2x+19)$. Since $x^2+2x+19$ has no real roots, it follows that $s+t=2$.

Generalized Solution I by Donald P. Skow, University of Texas-Pan American, Edinburg, Texas.

Let $x \geq 0$ and define

$$f(x) = \sqrt[3]{(3x+1) + (x+3)\sqrt{x}} + \sqrt[3]{(3x+1) - (x+3)\sqrt{x}}.$$

Then $f(6) = \sqrt[3]{19 + 9\sqrt{6}} + \sqrt[3]{19 - 9\sqrt{6}}$. But $f(6) = 2$ since

$$f(x) = \sqrt[3]{(1+\sqrt{x})^3} + \sqrt[3]{(1-\sqrt{x})^3} = 2$$

for all x.

Generalized Solution II by Joseph E. Chance, University of Texas-Pan American, Edinburg, Texas.

If

$$S = \sqrt[3]{a + c\sqrt{b}} + \sqrt[3]{a - c\sqrt{b}} = s + t,$$

then

$$(*) \qquad S^3 - 3\sqrt[3]{a^2 - c^2 b} \cdot S - 2a = 0.$$

If we let $a^2 - c^2 b = w^3$, then other representations of this form for 2 can be found if 2 is a root of $(*)$, or

$$w = \frac{4-a}{3}.$$

Some experimenting yields at least two other choices,

$$\sqrt[3]{31 + 13\sqrt{10}} + \sqrt[3]{31 - 13\sqrt{10}} = 2, \quad \text{and}$$

$$\sqrt[3]{37 + 30\sqrt{3}} + \sqrt[3]{37 - 30\sqrt{3}} = 2.$$

54. [1993, 39] *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Let $x_1 = 1$ and

$$x_j - x_{j-1} = \sum_{k=1}^{j} (-1)^{k+1} \binom{j}{k} \sum_{m=1}^{k} \frac{1}{m}, \quad \text{for } j \geq 2.$$

Prove that for any positive integer $n > 1$,

$$\sum_{k=1}^{n} k \left(k x_k^{-1}\right)^k < (n+1)!\ .$$

Solution by the proposer.

From

$$\binom{j}{k} = \binom{j-1}{k} + \binom{j-1}{k-1}$$

and some simple calculations we get

$$\sum_{k=1}^{j} (-1)^{k+1} \binom{j}{k} \sum_{m=1}^{k} \frac{1}{m} = \frac{1}{j}.$$

(A good reference on this topic is John Riordan, *Combinatorial Identities,* John Wiley & Sons, Inc., New York, 1968, p. 5.) Hence, $x_j - x_{j-1} = 1/j$, and thus

$$x_n = \sum_{j=1}^{n} \frac{1}{j}.$$

Now, from the relationship between the arithmetic and the geometric means of the numbers $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}$, we deduce that

$$\frac{n^n}{n!} < \left(\sum_{j=1}^{n} \frac{1}{j}\right)^n = x_n^n.$$

Therefore, $n! > (nx_n^{-1})^n$. This implies that $n(n!) > n(nx_n^{-1})^n$. Consequently,

$$\sum_{k=1}^{n} k(kx_k^{-1})^k < \sum_{k=1}^{n} k(k!) = (n+1)! - 1 < (n+1)!\ .$$

55. [1993, 40] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Let F_n and L_n denote the nth Fibonacci and Lucas numbers, respectively. Find a polynomial $f(x, y)$ with constant coefficients such that $f(F_n, L_n)$ is identically zero for all positive integers n or prove that no such polynomial exists.

Solution by the proposer.

From problem 35 in the *Missouri Journal of Mathematical Sciences* [1992, 96–97],

$$4(-1)^n = L_n^2 - 5F_n^2.$$

Square both sides to get terms with constant coefficients. The desired polynomial is therefore

$$f(x, y) = 25x^4 - 10x^2y^2 + y^4 - 16.$$

Generalized Solution by Joseph E. Chance, University of Texas-Pan American.

Let A_n and B_n be any two solutions of the difference equation

$$X_{n+1} = aX_n + bX_{n-1},$$

written as

$$A_n = C_1\alpha^n + C_2\beta^n$$
$$B_n = D_1\alpha^n + D_2\beta^n$$

where α, β solve $x^2 - ax - b = 0$, $\alpha \neq \beta$. Using Cramer's rule, it follows that

$$\alpha^n = \frac{D_2A_n - C_2B_n}{D}$$

and

$$\beta^n = \frac{C_1B_n - D_1A_n}{D},$$

where we assume that

$$D = \det\begin{pmatrix} C_1 & C_2 \\ D_1 & D_2 \end{pmatrix} \neq 0.$$

Thus

$$(C_1B_n - D_1A_n)(C_2B_n - D_2A_n) = -D^2(\alpha\beta)^n = -D^2(-b)^n.$$

This product is dependent upon n unless $b = -1$. If $b = 1$, the product can be squared to remove the dependency on n, as in the case of Fibonacci and Lucas numbers. For other values of b, the product is dependent on n and this technique fails to suggest an appropriate polynomial.

56. [1993, 40] *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

Let

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$A_{n+1} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & & & & & \\ 0 & 1 & & & & & \\ 0 & 0 & & & A_n & & \\ \vdots & \vdots & & & & & \\ 0 & 0 & & & & & \end{pmatrix}$$

for $n \geq 1$. Find $\det A_{1993}$.

Solution by Seung-Jin Bang, Albany, California.

We will prove that $\det A_n = -\det A_{n-3}$. Replacing the 3rd row by the 3rd row minus the 2nd row and the 4th row by the 4th row minus the 1st row, we have

$$\det \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & & & & & \\ 0 & 0 & 0 & 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & A_{n-3} & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & & & \end{pmatrix}$$

Next, we expand the determinant by the 3rd and 4th rows. Then

$$\det\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & & & A_{n-3} & & \\ \vdots & \vdots & \vdots & \vdots & & & & & \\ 0 & 0 & 0 & 0 & & & & & \end{pmatrix}$$

and then expand about the 1st and 2nd columns. Then we have

$$-\det\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & & & \\ 0 & 0 & & & & & \\ 0 & 0 & & & A_{n-3} & & \\ \vdots & \vdots & & & & & \\ 0 & 0 & & & & & \end{pmatrix} = -\det A_{n-3}.$$

Since $1993 = 3 \cdot 664 + 1$, we have $\det A_{1993} = \det A_1 = -1$.

Also solved by the proposers.