A PERPLEXING FINITE CONTINUED FRACTION

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Let

$$H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

be the nth Harmonic number. Although there is no closed form expression for H_n in terms of common mathematical functions, it is known that H_n *can* be expressed in terms of Stirling numbers. Specifically,

$$H_n = \frac{1}{n!}\begin{bmatrix} n+1 \\ 2 \end{bmatrix}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes the Stirling Cycle number (Stirling number of the first kind) which is equal to the number of permutations of n objects that have exactly k cycles ([4], p. 261).

In the early 1980's, I became interested in the related expression,

$$S_n = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{\ddots + \cfrac{1}{n}}}}}$$

also written as

$$S_n = \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\cdots + \frac{1}{n}$$

which is a simple finite continued fraction with n terms. Specifically, I wanted to know if there is a closed form expression for S_n, and if not, can S_n be expressed in terms of H_n?

The sequence S_n begins:

$$\frac{1}{1}, \frac{2}{3}, \frac{7}{10}, \frac{30}{43}, \frac{157}{225}, \frac{972}{1393}, \frac{6961}{9976}, \frac{56660}{81201}, \frac{516901}{740785}, \frac{5225670}{7489051}, \;\cdots\;.$$

It is not hard to show that the numerators and the denominators both satisfy the recurrence:

$$(1) \qquad\qquad u_n = nu_{n-1} + u_{n-2}, \qquad n \geq 3$$

(see [5], p. 130) and all the fractions are in lowest terms. The sequence S_n represents the convergents of the corresponding infinite continued fraction. Let p_n be the sequence of numerators and q_n the sequence of denominators. We have the obvious initial conditions: $p_1 = 1$, $p_2 = 2$ and $q_1 = 1$, $q_2 = 3$. We can also define $p_0 = 0$ and $q_0 = 1$ and still satisfy the recurrence.

I was unable to solve this recurrence, so in 1984 I posted the recurrence (1) in the "Math Notes Conference", a computer bulletin board at Digital Equipment Corporation, and asked for help.

Almost immediately, Peter Gilbert (at DEC) responded by showing how to form a corresponding differential equation. This turns out to be a standard method for solving such recurrences known as the method of generating functions (see chapter 7 of [4]), which we describe below.

Let us consider only the sequence of denominators, q_i (the numerators are similar). We define the function $f(x)$ by the formal power series

$$(2) \qquad\qquad f(x) = q_0 + q_1 x + q_2 x^2 + q_3 x^3 + \cdots + q_n x^n + \cdots$$

and then by a sequence of manipulations we use the recurrence to give us a difference equation or a differential equation for $f(x)$. We then solve this equation for $f(x)$ and upon expanding $f(x)$ out as a Taylor series, we then wind up with the sought-for coefficients q_i. (This technique frequently works even when the formal power series does not converge.) In this case, we proceed as follows. Start by taking the formal derivative of f.

$$f'(x) = q_1 + 2q_2 x + 3q_3 x^2 + 4q_4 x^3 + \cdots + nq_n x^{n-1} + \cdots .$$

Then multiply by the appropriate powers of x to get the coefficients of x^n to be the terms of our recurrence.

$$xf(x) = q_0 x + q_1 x^2 + \ q_2 x^3 + \ q_3 x^4 + \ q_4 x^5 + \cdots + q_{n-1} x^n + \cdots$$

$$x^2 f(x) = \qquad q_0 x^2 + \ q_1 x^3 + \ q_2 x^4 + \ q_3 x^5 + \cdots + q_{n-2} x^n + \cdots$$

$$x^2 f'(x) = \qquad q_1 x^2 + 2q_2 x^3 + 3q_3 x^4 + 4q_4 x^5 + \cdots + (n-1)q_{n-1} x^n + \cdots$$

Adding these three equations and then comparing with equation (2) gives

$$xf(x) + x^2 f(x) + x^2 f'(x) = f(x) - 1$$

or letting $y = f(x)$, we get the differential equation

$$x^2 y' + y(x^2 + x - 1) = -1.$$

Now the way I was taught to solve a non-homogeneous differential equation such as this is by first solving the corresponding homogeneous differential equation

$$x^2 y' + y(x^2 + x - 1) = 0.$$

We used Vaxima, running on a VAX, to solve this differential equation. We found that the solution is

(3) $$y = \frac{c}{x e^{x + 1/x}}.$$

If I try to expand out the right-hand side of equation (3) in a power series, both Vaxima and Mathematica fail, claiming that the function has an essential singularity at the origin. Basil Rennie (personal correspondence) informed us that the solution to the first differential equation is

$$y = -\frac{1}{x} \exp(-x - \frac{1}{x}) \int^x \frac{1}{t} \exp(t + \frac{1}{t}) \, dt$$

and that all solutions have an essential singularity at $x = 0$. What does this mean in terms of solving our recurrence equation? Does it mean that the recurrence has no closed form or merely that its solution has no generating function of the form (2)? Perhaps this approach is incorrect and a totally different approach should be used.

In 1988, in response to a request from Joe Keane (at Carnegie Mellon University) for interesting recurrences, I posted the recurrence (1) in the sci.math USENET newsgroup (an international computer bulletin board shared by various UNIX systems).

This elicited a reply from David desJardins (at UC Berkeley) who showed (with help from Macsyma) that

$$(4) \qquad q_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}^2 (n-2k)!$$

and

$$p_n = q_n - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} \binom{n-k}{k+2}\binom{n-k-2}{k}(n-2k-2)! \ .$$

An amazing result, but still not what I wanted.

Keane then posted the formula

$$(5) \qquad u(n) = (-1)^n \sum_{k=0}^{\infty} \frac{1}{k!(k+n+1)!}$$

stating that $u(n)$ solved the recurrence (1), but he gave no explanation.

Gaston Gonnet (at University of Waterloo) then posted an asymptotic expansion (derived using Maple). Subsequently, Gonnet and Gerald A. Edgar (at The Ohio State University) noted that

$$(6) \qquad i^n J_n(2i)$$

satisfies the recurrence (1) where $J_n(z)$ is the Bessel function (of the first kind) of order n.

This was getting close, but not quite right, since this solution did not satisfy the initial conditions for either p_n or q_n.

I then tried an exponential generating function. Letting

$$f(x) = q_0 + q_1 \frac{x}{1!} + q_2 \frac{x^2}{2!} + q_3 \frac{x^3}{3!} + \cdots + q_n \frac{x^n}{n!} + \cdots$$

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I was able to obtain the differential equation

$$(x-1)y'' + 2y' + y = 0.$$

However, this differential equation also does not seem to have a solution in terms of elementary functions.

This was getting me nowhere.

So at this point, I sent the problem of expressing S_n in terms of H_n to the American Mathematical Monthly problem column, hoping that some reader would come up with a solution. However, the Monthly problem column editors revised the problem and wound up printing it as a request for finding asymptotic estimates for p_n and q_n [8]. Furthermore, they omitted all mention of H_n.

Fortunately, the Monthly solvers lived up to their reputation and gave more than was asked for. Gerald Edgar and Klaus Zacharias noted [2] that if K_n and I_n denote the usual modified Bessel functions (see [10], pp. 79–80, or chapter 9 of [1]), which can be defined (for positive integral n and complex z) by the series

(7) $$I_n(z) = \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!(n+k)!}$$

and

$$K_n(z) = \frac{1}{2}\left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\left(-z^2/4\right)^k + (-1)^{n+1} \ln\left(\frac{z}{2}\right) I_n(z)$$

$$+ (-1)^n \frac{1}{2}\left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \{\psi(k+1) + \psi(n+k+1)\} \frac{(z^2/4)^k}{k!(n+k)!}$$

where

$$\psi(n) = -\gamma + H_{n-1},$$

then we have the following recurrences:

$$K_{n+1}(x) = \frac{2n}{x} K_n(x) + K_{n-1}(x)$$

$$I_{n+1}(x) = \frac{-2n}{x} I_n(x) + I_{n-1}(x).$$

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Thus, $K_{n+1}(2)$ and $(-1)^{n+1}I_{n+1}(2)$ both satisfy the recurrence (1). This agrees with (6) since it is known that

$$I_n(z) = i^{-n}J_n(iz).$$

It also shows why equation (5) holds, since the series in (5) is a special case of equation (7) and represents $(-1)^{n+1}I_{n+1}(2)$.

Since any linear combination of the solutions $K_{n+1}(2)$ and $(-1)^{n+1}I_{n+1}(2)$ is also a solution to the recurrence, we can find multipliers to get the initial conditions correct. Using

$$K_{n+1}(x)I_n(x) + K_n(x)I_{n+1}(x) = \frac{1}{x}$$

we thus find that

$$p_n = 2(-1)^{n+1}K_1(2)I_{n+1}(2) + 2I_1(2)K_{n+1}(2)$$

and

$$q_n = 2(-1)^n K_0(2)I_{n+1}(2) + 2I_0(2)K_{n+1}(2).$$

This is an explicit solution to my original problem, however, I did not have Bessel functions in mind as a "closed form" for S_n.

It is interesting to note (see [1]) that generating functions for $I_n(z)$ and $J_n(z)$ are given by

$$e^{\frac{1}{2}z(t+1/t)} = \sum_{n=-\infty}^{\infty} t^n I_n(z) \quad \text{and} \quad e^{\frac{1}{2}z(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(z), \quad (t \neq 0).$$

At about the same time, D. H. Fowler [3] asked in *The Mathematical Intelligencer* for the exact value of the infinite continued fraction

$$S = \frac{1}{1+} \frac{1}{2+} \frac{1}{3+} \cdots = [1, 2, 3, \ldots] .$$

In the published solution, M. S. Klamkin stated that the result

(8)
$$[n, n+d, n+2d, n+3d, \ldots] = \frac{I_{\frac{n}{d}-1}\left(\frac{2}{d}\right)}{I_{\frac{n}{d}}\left(\frac{2}{d}\right)}$$

follows easily from the material in [6]. In particular, we have

$$[n, n+1, n+2, n+3, \ldots] = \frac{I_{n-1}(2)}{I_n(2)}.$$

This generalizes AMM problem 4631 (see [7]), where it was shown that $[1, 2, 3, \ldots] = I_0(2)/I_1(2)$.

From equation (8) we see that the continued fraction $[k, 3k, 5k, \ldots]$ can be put in the closed form

$$[k, 3k, 5k, \ldots] = \frac{I_{-1/2}(1/k)}{I_{1/2}(1/k)}.$$

Amazingly, this can be simplified by using the identities

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cosh x \qquad \text{and} \qquad I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x$$

from [1] to yield

$$[k, 3k, 5k, \ldots] = \coth \frac{1}{k}$$

which is a well-known continued-fraction expansion for coth.

Does the fact that this result can be expressed in terms of hyperbolic trigonometric functions give any hope to our being able to express related results in terms of trigonometric expressions?

Open Questions.

Getting back to S_n, we leave the reader with the following open questions:

1. Is there a simple closed form for S_n not involving Bessel functions or Hypergeometric functions?
2. Can S_n, p_n, or q_n be expressed in terms of Harmonic numbers? (how about Stirling numbers?)
3. Is there a polynomial $g(x, y, z)$, such that $g(H_n, S_n, n) = 0$ for all positive integers n?

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