DECOMPOSITION OF THE LINE INTO COUNTABLY-MANY MEASURE-THEORETIC DENSE SETS

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Like the warp and woof of a piece of cloth, two sets may be thoroughly intermingled. But how intermingled can disjoint sets be? With this in mind we ask the following question:

> Can $\mathbb{R}^n$ be decomposed into countably-many (or even just two) disjoint Lebesgue measurable sets such that the intersection of any one of these sets with *any* continuous (non-constant) curve has positive one-dimensional Hausdorff measure (or, at least, positive Hausdorff dimension)? (For the definition of Hausdorff measure and Hausdorff dimension, see, e.g., [1].)

In this note we show that the one-dimensional case is true, that is, we show that the real line $\mathbb{R}$ can be decomposed into countably-many Lebesgue measurable sets such that the intersection of any of these sets with *any* open interval has positive measure. (Decomposition of $\mathbb{R}$ into two such sets was posed as a problem in [2, p. 59].)

We call a measurable subset E of an interval I an m-dense set (with respect to I) if for any open interval $I_1 \subset I$ we have $0 < m(E \cap I_1)$, where m represents one-dimensional Lebesgue measure. The one-dimensional problem is then whether $\mathbb{R}$ can be expressed as the union of mutually-disjoint m-dense sets. It suffices to carry out the construction of countably-many m-dense sets on the unit interval $[0, 1)$, then extend these sets to be periodic of period one. Suppose the following statement is true:

(1) Any set B which is m-dense with respect to the unit interval is the union of two disjoint m-dense sets C and D with $m(D) = m(B)/2$.

Then, starting with the unit interval and iterating this decomposition, we obtain

$$[0, 1) = \bigcup_{1}^{n} A_i \cup B_n$$

for all $n \geq 1$, where $B_0 = [0,1)$, and $B_n, n \geq 0$, is the union of disjoint m-dense sets A_{n+1} and B_{n+1} such that $m(B_{n+1}) = m(B_n)/2$. Then, $[0,1) = \cup_1^\infty A_i \cup B_\infty$, where B_∞ has measure 0 and so can be incorporated into any of the other sets. Thus, it suffices to prove statement (1).

First, given a set B of positive measure contained in an interval I of length L, we define a generalized Cantor set $E = E(B, I, \mu), 0 < \mu < 1$. Given $\delta, 0 < \delta < L$, choose $\{\delta_n\}$ so that

$$L = \delta_0 > \delta_1 > \delta_2 > \cdots, \delta_n \to \delta. \tag{2}$$

Put $E_0 = \bar{I}$. For $n \geq 0, E_n$ is constructed so that E_n is the union of 2^n disjoint closed intervals, each of length $2^{-n}\delta_n$. Delete an open interval in the center of each of these 2^n intervals, so that each of the remaining 2^{n+1} intervals has length $2^{-n-1}\delta_{n+1}$ and let E_{n+1} be the union of these 2^{n+1} intervals. Then $E_1 \supset E_2 \supset \cdots, m(E_n) = \delta_n$, and the generalized Cantor set $E = \cap_1^\infty E_n$ has measure δ. Here δ is chosen so that $m(E \cap B) = (1-\mu)m(B)$ (a continuity argument shows that this can be done). Thus, $F = F(B, I, \mu)$, the complement of E with respect to I, is a dense open subset of I satisfying $m(F \cap B) = \mu m(B)$. (Note that E and F depend on the choice of the sequence $\{\delta_n\}$.)

We now give the proof of statement (1). Choose positive numbers $\mu_i < 1, i = 1, 2, \ldots,$ such that $\prod_1^\infty \mu_i = 1/2$. We define a decreasing sequence of open dense sets D_n contained in the unit interval such that

$$m(B \cap D_n) = \prod_1^n \mu_i \cdot m(B). \tag{3}$$

The set $D_1 = F(B, [0,1), \mu_1)$ is the union of disjoint open intervals $I_j^{(1)}, j \geq 1$. Let

$$D_2 = \bigcup_{j=1}^\infty F(B \cap I_j^{(1)}, I_j^{(1)}, \mu_2).$$

Then, equation (3) is satisfied for $n = 2$, and $D_2 \subset D_1$. D_2 is open so it is also the union of disjoint open intervals $I_j^{(2)}$. Continuing this process we obtain the prescribed sequence of open sets D_n.

Let $D = B \cap (\cap_1^\infty D_n)$. We have $m(D) = \lim m(D_n \cap B) = m(B)/2$, since $D_n \cap B$ form a decreasing sequence. Next, we show that the sets $C = B \backslash D$ and D are m-dense subsets of the unit interval provided that in (2) the term δ_1 is always chosen larger than $L/2$. With this proviso the subintervals $I_j^{(n)}$ of D_n have length less than 2^{-n}. Since $D_n, n \geq 1$, is a dense subset of $[0, 1)$, any interval $I \subset [0, 1)$ contains an interval of the form $I_j^{(n)}$ for n sufficiently large. But, by the construction,

$$m(D \cap I) \geq m(D \cap I_j^{(n)}) = \prod_{n+1}^{\infty} \mu_i \cdot m(B \cap I_j^{(n)}) > 0.$$

The above *equality* shows that $m(D \cap I_j^{(n)}) < m(I_j^{(n)})$ so that $m(C \cap I) > 0$. This completes the proof of statement (1). Note that the decomposition of the unit interval into m-dense sets A_i, the measure of $A_i, i \geq 2$, can be made arbitrarily small by taking $\prod \mu_i$ slightly less than 1. A simple argument then shows that the *real line* can be decomposed into m-dense sets, all but one of which has measure less than an arbitrarily assigned positive number.

References

1. K. J. Falconer, *The Geometry of Fractal Sets*. Cambridge University Press, Cambridge, 1985.

2. W. Rudin, *Real and Complex Analysis*, 2nd ed., McGraw-Hill Book Co., New York, 1974.