ANOTHER PROOF THAT THE CLOSED UNIT INTERVAL IS A CONTINUOUS IMAGE OF THE CANTOR SET

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It is well known ([1] for example) that the closed unit interval is the continuous image of the Cantor Set. The usual proof proceeds as follows.

The standard Cantor middle thirds set can be realized as the set

$$C = \left\{ \sum_{n=1}^{\infty} \frac{t_n}{3^n} : t_n = 0 \quad \text{or} \quad t_n = 2 \right\}.$$

One may then construct a map $f : C \to I$ by

$$f : \sum_{n=1}^{\infty} \frac{t_n}{3^n} \mapsto \sum_{n=1}^{\infty} \frac{\phi(t_n)}{2^n},$$

where $\phi(0) = 0$ and $\phi(2) = 1$. One then checks that f is continuous and surjective.

This is an elegant proof that cannot be improved upon.

That which follows is an alternate proof of the same result. The main virtue of this approach is that it introduces the reader to some classical results of general topology and to the concept of the code space which is very useful in the growing area of dynamical systems.

1. The Code Space. In the sequel, we will let S denote the set of all functions from N (the set of natural numbers $1, 2, 3, \dots$) to the set $\mathbb{Z}_2$ (the integers $\pmod 2$) which contains two elements denoted by 0 and 1). We will topologize S by describing a basis. For every $x \in S$ let $N(x, 0) = S$. For every $x \in S$ and $n \in \mathbb{N}$, let

$$N(x, n) = \{y \in S : y(i) = x(i), 1 \leq i \leq n\}.$$

It is clear that $\{N(x,n) : x \in S, n = 0,1,2,\dots\}$ is a base for a topology on S. We will always understand S to be equipped with this topology. The space S is often referred to as *the code space on the symbols 0 and 1*.

It turns out that S is a metric space in this topology. A function $d : S \times S \to [0,\infty)$ can be defined as follows. Let $d(x,y) = \sup\{\frac{1}{m} : y \in N(x, m-1), m \in \mathbb{N}\}$. It is left as an exercise for the reader that d is a metric that induces the given topology on S.

It is also useful to note that S is homeomorphic to the product space $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots$ with the product topology where $\mathbb{Z}_2$ is understood to have the discrete topology. It now follows by the Tychonoff Theorem that S is compact.

A topological space X is said to be *perfect* if every point in X is a limit point of X. A space X is said to be *totally disconnected* if none of its connected subset contains more than one point.

We now state a classical theorem of general topology whose proof may be found in [1].

Theorem 1. If X is a compact, perfect, totally disconnected metric space, then X is homeomorphic to the Cantor Set.

We may now prove the following.

Corollary 2. S is homeomorphic to the Cantor Set.

Proof. As we have already stated, S is a compact metric space. It is easy to check that S is perfect. It suffices to show that S is totally disconnected.

Now suppose that T is a subset of S that contains points $x \neq y$. We wish to construct a separation of T. Let $k \in \mathbb{N}$ be such that $x(k) \neq y(k)$. Of course $N(x,k)$ is open in S; we claim that $S - N(x,k)$ is open in S. Suppose that $z \in S - N(x,k)$. Let $w \in N(z,k)$. Note that there is a $1 \leq i \leq k$ such that $z(i) \neq x(i)$. Hence, $w(i) \neq x(i)$. It now follows that $N(z,k) \subset S - N(x,k)$. Therefore $S - N(x,k)$ is open.

Let $U = N(x,k) \cap T$ and let $V = (S - N(x,k)) \cap T$. Since $x \in U$ and $y \in V$ it follows that (U,V) is a separation for T.

Observe that any subset of a totally disconnected space is totally disconnected and any subset of a metric space is metric. Therefore we have the following.

Corollary 3. A closed, perfect subset of a set homeomorphic to the Cantor Set is homeomorphic to the Cantor Set.

2. Special Subsets of S. Let $x \in S$. If $x(n) = x(n+1) = \dots = x(n+k-1)$ for some $n,k \in \mathbb{N}$, then we say that x has a *run of length* k. Define $\rho(x) = \sup\{k : x$ has a run of length $k\}$.

For $n \in \mathbb{N}$, let $B_n = \{x \in S : \rho(x) \leq n\}$.

Lemma 4. B_n is closed for every $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$. Suppose that $x \in S - B_n$. Then x has a run of length $n+1$. In particular, say that $x(l) = x(l+1) = \cdots = x(l+n)$ for some $l \in \mathbb{N}$. Note that $N(x, l+n) \subset S - B_n$. Therefore B_n is closed.

Lemma 5. If $n \geq 2$, then B_n is perfect.

Proof. Let $n \geq 2$ be given. Suppose that $x \in B_n$. We claim that x is a limit point of B_n. Let $k \in \mathbb{N}$. We wish to show that $N(x,k)$ contains an element y of B_n besides x.

Let $y(i) = x(i)$ for $1 \leq i \leq k$. Suppose that $x(k) = 0$. If $x(k+l) = 0$ for all even $l \in \mathbb{N}$ and $x(k+l) = 1$ for all odd $l \in \mathbb{N}$, then define $y(k+1) = 1$ and $y(k+2) = 1$ and define $y(k+l) = 0$ for all odd $l \in \mathbb{N}$ with $l \geq 3$ and $y(k+l) = 1$ for all remaining $l \in \mathbb{N}$; otherwise define $y(k+l) = 0$ for all even $l \in \mathbb{N}$ and $y(k+l) = 1$ for all odd $l \in \mathbb{N}$. Then $y \neq x$ and $y \in B_n$. In the case that $x(k) = 1$, we may proceed similarly.

Theorem 6. If $n \geq 2$, then B_n is homeomorphic to the Cantor Set.

Proof. This follows by Corollary 3 and Lemmas 4 and 5.

3. A Special Mapping. For $m, n \in \mathbb{N}$ with $m \leq n$, let $[m,n] = \{k \in \mathbb{N} : m \leq k \leq n\}$.

Let $n \in \mathbb{N}$ and $x \in B_n$ be given. Suppose that

$$x(m) = x(m+1) = \cdots = x(m+k-1) \neq x(m+k)$$

and either $m = 1$ or $x(m-1) \neq x(m)$. Then we say that $[m, m+k-1]$ is a segment of x, that is of length k.

Note that the segments of $x \in S$ form a partition of $\mathbb{N}$. Given $x \in S$ let $\{[a_k, b_k] : k \in \mathbb{N}\}$ be the set of segments of x with notation chosen so that $a_1 = 1$ and $a_{k+1} = b_k + 1$ for all $k \in \mathbb{N}$. Let $\delta(x,k) = b_k - a_k$. Note that $\delta(x,k)+1$ is the length of $[a_k, b_k]$.

Define $g : B_{10} \to I$ by

$$g(x) = \sum_{k=1}^{\infty} \frac{\delta(x,k)}{10^k}.$$

Theorem 7. g is a continuous surjection.

Proof. Let $x \in B_{10}$ be given. Let $\{[a_k, b_k] : k \in \mathbb{N}\}$ be the set of segments of x with notation chosen as above. Suppose that $n \geq b_k$ and $y \in N(x, n)$. Then $\delta(x, i) = \delta(y, i)$ for $1 \leq i \leq k$. So

$$|g(x) - g(y)| \leq \sum_{j=k+1}^{\infty} \left(\frac{9}{10}\right)^j \leq \left(\frac{1}{10}\right)^k.$$

It follows that g is continuous.

To show that g is a surjection, let $a \in I$. Then

$$a = \sum_{i=1}^{\infty} \frac{a^i}{10^i}$$

for some $a_i \in \mathbb{N}$ and $0 \leq a_i \leq 9$. One may easily create an $x \in B_{10}$ such that $\delta(x, i) = a_i$ for all $i \in \mathbb{N}$.

Reference

1. J. G. Hocking and G. S. Young, *Topology*, Dover, 1988.