A CHALLENGING AREA PROBLEM REVISITED

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Some years ago a problem was proposed in the *American Mathematical Monthly* [1] for which the editors received no correct solutions before the deadline. Although eventually a solution was published (under the title "One Tough Area Problem" [2]), it is relatively involved. I would like to present a quite different, simpler solution.

The problem is to find the area of the convex planar region

$$R = \{P : PA + PB + PC \leq 2a\},$$

where ABC is an equilateral triangle of perimeter $3a$.

For convenience we take $a = 1$. We start by imposing a rectangular coordinate system in which the coordinates of A, B, C are $(-1/2, 0)$, $(1/2, 0)$, and $(0, \sqrt{3}/2)$ respectively. As mentioned in [2], the convexity of R is relatively easy to show using the triangle inequality. Let ∂R denote the boundary of R. Clearly A, B and C are on ∂R. We may deduce that the portion of ∂R in quadrant I is a convex curve connecting C and B. A parameterization of this curve may be obtained by constructing a circle of radius r, $0 \leq r \leq 1$, with center C; and an ellipse with foci A and B, and major axis of length $2 - r$. If P is the point of intersection in quadrant I, then $PC = r$ and $PA + PB = 2 - r$, so that $PA + PB + PC = 2$. See the figure below. As r goes from 0 to 1, P travels along ∂R from C to B. The coordinates (x, y) of P can be found by solving the system

$$x^2 + (y - \sqrt{3}/2)^2 = r^2 \tag{1}$$

$$\frac{x^2}{\left(\frac{2-r}{2}\right)^2} + \frac{y^2}{\left(\frac{2-r}{2}\right)^2 - \left(\frac{1}{2}\right)^2} = 1. \tag{2}$$

Multiplying (1) by $\left(\frac{2-r}{2}\right)^{-2}$ and subtracting the result from (2) we eliminate x^2, and find (after a bit of algebra) that $y = (1-r)\sqrt{3-r}(2 - \frac{\sqrt{3}}{2}\sqrt{3-r})$. Setting

$$s = \sqrt{3-r} \tag{3}$$

we obtain

$$y = s(s^2-2)\left(2 - \frac{\sqrt{3}}{2}s\right). \tag{4}$$

From (1) we get $x^2 = r^2 - (y - \sqrt{3}/2)^2 = (r - y + \sqrt{3}/2)(r + y - \sqrt{3}/2)$. Therefore by (3) and (4) x^2 equals

$$\left[(s^2-3) - s(s^2-2)\left(2 - \frac{\sqrt{3}}{2}s\right) + \frac{\sqrt{3}}{2}\right]\left[(s^2-3) + s(s^2-2)\left(2 - \frac{\sqrt{3}}{2}s\right) - \frac{\sqrt{3}}{2}\right]. \tag{5}$$

We may make use of the system (1)–(2) to factor (5) completely. From equation (1) we see that $r = 0$ (and hence $s = \sqrt{3}$) implies that $x = 0$. We find that $\sqrt{3}$ is a root of both factors in (5). In our search for other roots we note that by (2) $r = 2$ (and hence $s = 1$) also implies that $x = 0$. We verify that 1 is a double root of the first factor in (5). Next we observe that in (4) $s = 1$ gives $y = -2 + \sqrt{3}/2$. But, when $r = 2$, from (1) $y = 2 + \sqrt{3}/2$ is also a possibility. This value may be obtained in (4) by setting $s = -1$. We verify that $s = -1$ is a double root of the second factor in (5). From this we may factor (5):

$$x^2 = \frac{3}{4}(s - \sqrt{3})^2(s-1)^2(s+1)^2\left[4 - \left(s - \frac{\sqrt{3}}{3}\right)^2\right].$$

Setting $2t = s - \sqrt{3}/3$, and simplifying we arrive at the following parameterization for the portion of ∂R in the first quadrant:

$$x(t) = \left(-8\sqrt{3}t^3 + 4\sqrt{3}t - \frac{4}{3}\right)\sqrt{1-t^2} \tag{6}$$

$$y(t) = -8\sqrt{3}t^4 + 8\sqrt{3}t^2 - \frac{4}{3}t - \frac{5\sqrt{3}}{6} \tag{7}$$

where $r \in [0,1] \implies s \in [\sqrt{2}, \sqrt{3}] \implies t \in [(\sqrt{2}-\sqrt{3}/3)/2, \sqrt{3}/3]$.

From calculus,

$$\alpha := \text{area of } R \text{ in quadrant I} = \int_{(\sqrt{2}-\sqrt{3}/3)/2}^{\sqrt{3}/3} x(t)y'(t)\,dt.$$

Once we determine the value of α the computation of area(R) follows. We may decompose R into the equilateral triangle ABC, and three regions each congruent to the region of R in quadrant I outside of the segment CB. This latter region has area $\alpha - \sqrt{3}/8$ and hence

$$\text{area}(R) = \text{area}(\triangle ABC) + 3(\alpha - \sqrt{3}/8) = 3\alpha - \sqrt{3}/8. \tag{8}$$

From (6) and (7) we obtain

$$x(t)y'(t) = 16\left(48t^6 - 48t^4 + \frac{10\sqrt{3}}{3}t^3 + 12t^2 - \frac{5\sqrt{3}}{3}t + \frac{1}{9}\right)\sqrt{1-t^2}. \tag{9}$$

The integrals $I_n := \int t^n \sqrt{1-t^2}\,dt$; $n = 0,1,2,3,4,6$; are standard and are given by

$$I_0 = \frac{1}{2}t\sqrt{1-t^2} + \frac{1}{2}\sin^{-1} t$$

$$I_1 = \frac{1}{3}(-1+t^2)\sqrt{1-t^2}$$

$$I_2 = \frac{1}{4}\left(-\frac{1}{2}t + t^3\right)\sqrt{1-t^2} + \frac{1}{8}\sin^{-1} t$$

$$I_3 = \frac{1}{5}\left(-\frac{2}{3} - \frac{1}{3}t^2 + t^4\right)\sqrt{1-t^2}$$

$$I_4 = \frac{1}{6}\left(-\frac{3}{8}t - \frac{1}{4}t^3 + t^5\right)\sqrt{1-t^2} + \frac{1}{16}\sin^{-1} t$$

$$I_6 = \frac{1}{8}\left(-\frac{5}{16}t - \frac{5}{24}t^3 - \frac{1}{6}t^5 + t^7\right)\sqrt{1-t^2} + \frac{5}{128}\sin^{-1} t.$$

Multiplying each of the I_n by the appropriate coefficient from (9) and combining terms we obtain

$$\alpha = \left[16\left\{\left(\frac{\sqrt{3}}{9}-\frac{23}{72}t-\frac{7\sqrt{3}}{9}t^2+\frac{15}{4}t^3+\frac{2\sqrt{3}}{3}t^4-9t^5+6t^7\right)\sqrt{1-t^2}+\frac{31}{72}\sin^{-1}t\right\}\right]_{(\sqrt{2}-\sqrt{3}/3)/2}^{\sqrt{3}/3}.$$

By some tedious but straightforward computations we may evaluate this last expression, and using (8), we find that the desired area is

$$-\frac{10\sqrt{2}}{9}+\left(\frac{187}{72}-\frac{89\sqrt{6}}{72}\right)\sqrt{5+\sqrt{24}}+\frac{62}{3}\left[\sin^{-1}\left(\frac{\sqrt{3}}{3}\right)-\sin^{-1}\left(\frac{\sqrt{2}-\sqrt{3}/3}{2}\right)\right]-\frac{\sqrt{3}}{8}.$$

References

1. E 2983, *American Mathematical Monthly,* 90 (1983), 54.
2. *American Mathematical Monthly,* 96 (1989), 642–645.

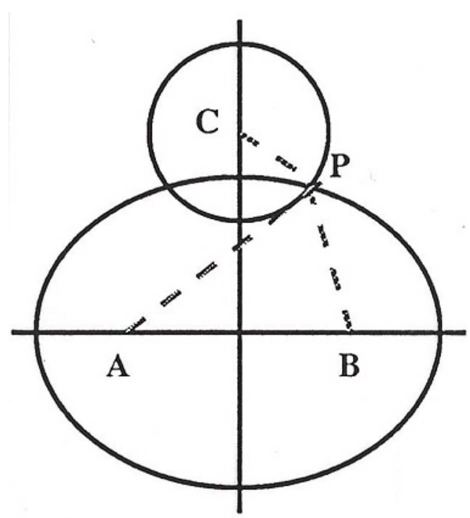

Figure.