THERE MAY BE MORE THAN ONE WAY TO FIND THE DERIVATIVE OF A FUNCTION

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Students with experience in integration techniques know that some integrals may be integrated in more than one way; for example, by means of different substitutions, usage of different trigonometric and algebraic identities, etc. However, most students have the perception that there is only one way to differentiate a given expression. The purpose of this article is to show, indeed, that the derivative of a function may also be found in more than one way. Although we are going to look at one particular function in this article, we hope that this will be a motivation for the reader to investigate and find other functions that can be differentiated in several ways.

Let

$$f(x) = (a_1x + b_1)^{m_1}(a_2x + b_2)^{m_2} \cdots (a_nx + b_n)^{m_n},$$

where a_i, b_i are arbitrary complex numbers, and m_i are arbitrary real numbers $(i = 1, 2, \ldots, n)$. For the sake of simplicity we write

$$f(x) = A(x + B_1)^{m_1}(x + B_2)^{m_2} \cdots (x + B_n)^{m_n},$$

where

$$A = a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n} \quad \text{and} \quad B_i = \frac{b_i}{a_i}.$$

Problem. If

$$f(x) = (a_1x + b_1)^{m_1}(a_2x + b_2)^{m_2} \cdots (a_nx + b_n)^{m_n}, \tag{1}$$

then

$$f'(x) = f(x)\left(\frac{m_1}{x + B_1} + \frac{m_2}{x + B_2} + \cdots + \frac{m_n}{x + B_n}\right), \tag{2}$$

where a_i, b_i, m_i, B_i $(i = 1, 2, \ldots, n)$, and A are as above.

Proof. We show in three different ways that (2) is the derivative of (1).

Method 1. Anyone with some knowledge of differentiation is familiar with this method. We apply the chain rule as well as the product rule.

Method 2. In this method we first take the logarithm of both sides of (1) and then we differentiate. Notice that for large n this method is quicker and therefore, preferable over the first method.

Method 3. Finally, we are going to obtain (2) from (1) without actually calculating $f'(x)$. To the author's knowledge this method is new, and was the motivation for writing this article. First note that

$$f(x+h) = f(x)\left(1+\frac{h}{x+B_1}\right)^{m_1}\left(1+\frac{h}{x+B_2}\right)^{m_2}\cdots\left(1+\frac{h}{x+B_n}\right)^{m_n}. \tag{3}$$

Next, let

$$g_i(h) = \left(1+\frac{h}{x+B_i}\right)^{m_i}, \quad i = 1, 2, \ldots, n.$$

Now, if m_i is a positive integer, then by the Binomial Theorem we have

$$g_i(h) = \sum_{k=0}^{m_i}\binom{m_i}{k}\left(\frac{h}{x+B_i}\right)^k. \tag{4}$$

Otherwise, the Maclaurin series for $g_i(h)$ is (for example, see [2, p. 329])

$$g_i(h) = 1 + \sum_{k=1}^{\infty}\frac{m_i(m_i-1)(m_i-2)\cdots(m_i-k+1)}{k!}\left(\frac{h}{x+B_i}\right)^k. \tag{5}$$

Thus, if for each i we substitute the expansion of $g_i(h)$ from (4) or (5) in (3), and then write the result in ascending powers of h, we get

$$f(x+h) = f(x) + f(x)\left(\frac{m_1}{x+B_1} + \frac{m_2}{x+B_2} + \cdots + \frac{m_n}{x+B_n}\right)h + \cdots . \tag{6}$$

Also, by Taylor's formula we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(x) + R_n. \tag{7}$$

Finally, comparing coefficients of h in expressions (6) and (7), again we obtain (2).

References

1. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions,* Dover Publications, Inc., New York, 1970.
2. R. P. Grimaldi, *Discrete and Combinatorial Mathematics, An Applied Introduction,* 2nd ed., Addison-Wesley Publishing Co., 1989.
3. R. E. Larson, R. P. Hostetler, and B. H. Edwards, *Calculus with Analytic Geometry,* 4th ed., D. C. Heath and Co., 1990.