SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

37. [1991, 149] *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Lines l_1 and l_2 are concurrent at O. Let $\{a_i\}$ be a sequence of points on l_1 and $\{b_i\}$ be a sequence of points on l_2 such that

$$d(O,a_1) = d(a_i,a_{i+1}) = d(O,b_1) = d(b_i,b_{i+1}) > 0$$

for $i = 1,2,3,\ldots$. If M_i is the midpoint of the line segment $\overline{a_ib_i}$, prove that the points M_i are collinear.

Solution I by Andrea Rothbart, Webster University, St. Louis, Missouri. This solution uses a vector approach. For simplicity, let $\overrightarrow{x}$ stand for the vector $\overrightarrow{Oa_1}$ and $\overrightarrow{y}$ stand for the vector $\overrightarrow{Ob_1}$. Then, $\overrightarrow{Ob_n} = n\overrightarrow{y}$ and $\overrightarrow{Oa_n} = n\overrightarrow{x}$. Also,

$$\overrightarrow{b_na_n} = \overrightarrow{b_nO} + \overrightarrow{Oa_n} = -n\overrightarrow{y} + n\overrightarrow{x} = n(\overrightarrow{x} - \overrightarrow{y})\ .$$

Thus,

$$\overrightarrow{OM_n} = \overrightarrow{Ob_n} + \overrightarrow{b_nM_n} = n\overrightarrow{y} + \frac{1}{2}n(\overrightarrow{x} - \overrightarrow{y}) = \frac{n}{2}(\overrightarrow{x} + \overrightarrow{y}) = n\overrightarrow{OM_1}\ .$$

Since each of the vectors $\overrightarrow{OM_n}$ is a multiple of $\overrightarrow{OM_1}$, the points M_1, M_2, $\ldots$ are collinear.

Solution II by Seung-Jin Bang, Seoul, Republic of Korea. We may assume that the line l_1 is the x-axis, the lines l_1 and l_2 are concurrent at $O = (0,0)$, and $d(O,a_1) = a > 0$. Then we have $a_i = (ia, 0)$.

Case I. The lines l_1 and l_2 are perpendicular. Then $b_i = (0, ia)$. It follows that

$$M_i = \frac{1}{2}(a_i + b_i) = \frac{ia}{2}(1,1)$$

and the points M_i lie on the line $y = x$.

Case II. The equation of the line l_2 is $y = kx$, $k = \tan\theta$ ($0 < \theta < \frac{\pi}{2}$, $\frac{\pi}{2} < \theta < \pi$). Then

$$b_i = (ia\cos\theta, ia\sin\theta)\ .$$

It follows that

$$M_i = \frac{1}{2}(a_i + b_i) = \frac{ia}{2}(1 + \cos\theta, \sin\theta)$$

and the points M_i lie on the line

$$y = \frac{\sin\theta}{1 + \cos\theta} x = \left(\tan\frac{\theta}{2}\right) x \ .$$

From Case I and II, the points M_i lie on the bisector of the angle $\angle a_1 O b_1$. This completes the proof.

Also solved by Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin and the proposer.

Andrea Rothbart noted that it is not necessary to assume $|\overrightarrow{Ob_1}| = |\overrightarrow{Oa_1}|$ *as stated in the problem.*

38. [1991, 149] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Consider the equation: $\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} = 0$. Bring the $\sqrt{x_3}$ term to the right-hand side and then square both sides. Then isolate the $\sqrt{x_1 x_2}$ term on one side and square again. The result is a polynomial and we say that we have rationalized the original equation.

Can the equation

$$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} = 0$$

be rationalized in a similar manner, by successive transpositions and squarings?

Solution by the proposer. Yes.

At first sight, it seems like as you square you will get more and more terms each time (when $n > 3$). However, proper grouping will in fact allow the equation to be rationalized.

First get rid of the radical containing x_n by bringing it to one side and then squaring both sides.

Now we will continue to get rid of the radical for each x_i in succession. Remove any square factors from within any radicals. Then find all radicals containing x_i. Bring them to the left-hand side and bring all the other terms to the right-hand side. The left-hand side is now of the form

$$\sqrt{x_i}\left(\sum_j a_j \sqrt{\prod_k x_{j_k}}\right)$$

where no j_k is i. Squaring both sides will now remove all the x_i terms from inside radicals. After at most n such operations, the equation will be rationalized.

One incorrect solution was received.

39. [1991, 150] *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

Let n be a positive integer and $L(i)$ denote the number of large digits (digits greater than or equal to 5) in the base 10 representation of the non-negative integer i. Evaluate

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} L(i)^4 .$$

Solution by Robert L. Doucette, McNeese State University, Lake Charles, Louisiana, Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin, and the proposers (independently).

Since $0 \le L(i) \le n$, for $0 \le i \le 10^n - 1$, we may rewrite the given expression as

$$\frac{1}{10^n}\sum_{m=0}^{n} \lambda_m m^4 ,$$

where λ_m denotes the number of elements in the set $\{i : L(i) = m\}$.

We may identify each i, $0 \le i \le 10^n - 1$, with an n-tuple in the set $\{0, 1, 2, \ldots, 9\}^n$. For each of $\binom{n}{m}$ choices of m positions, there are 5^m ways of filling them with large digits. The remaining $n - m$ positions can be filled in 5^{n-m} different ways with small digits. Hence,

$$\lambda_m = \binom{n}{m} 5^n .$$

Therefore, the given sum may be rewritten as

$$\frac{1}{2^n}\sum_{m=0}^{n} \binom{n}{m} m^4 .$$

By differentitating both sides of

$$(1+e^t)^n = \sum_{m=0}^{n} \binom{n}{m} e^{mt}$$

four times (with respect to t), and then evaluating at $t=0$, we have

$$\sum_{m=0}^{n} \binom{n}{m} m^4 = n(n+1)(n^2+5n-2)2^{n-4} \ .$$

Hence, the desired result is

$$\frac{1}{16}\big(n(n+1)(n^2+5n-2)\big) \ .$$

40. [1991, 150] *Proposed by Stan Wagon, Macalester College, St. Paul, Minnesota.*

A tetrahedron is a geometric solid with 4 vertices, 6 edges, and 4 triangular faces. A Heron triangle is one whose sides and area are integers. A Heron tetrahedron is one having Heron triangles as faces and whose volume is an integer.

(a) Show that if $\triangle ABC$ is acute, then a tetrahedron exists with each of its faces congruent to $\triangle ABC$.

(b)* John Leech has shown that a Heron tetrahedron exists: Let $\triangle ABC$ have sides 148, 195, and 203 and let T be the tetrahedron obtained from this triangle as in (a). Then each face of T has integer area and T has integer volume. The following question is inspired by Jim Buddenhagen's investigation of Heron triangles whose area is a square. Question: Is there a Heron tetrahedron whose volume is a perfect square or perfect cube?

Solution to (a) by the proposer. Place two copies of the triangle adjacent to each other along AB, so that a parallelogram $ACBC'$ is formed. Because the triangle is acute, $CC' > AB$; thus we may lift C and C' simultaneously, rotating the two triangles, until the distance between C and C' equals c. Connecting the moved points C and C' completes the tetrahedron. One can use integration of the cross-sections of the tetrahedron oriented this way – they are similar parallelograms – to obtain the following formula for the area of a tetrahedron having each face congruent to a triangle with side-lengths a, b, and c:

$$V = \sqrt{\frac{(a^2+b^2-c^2)(a^2+c^2-b^2)(b^2+c^2-a^2)}{72}} \ .$$

Applying this formula to Leech's example shows that the resulting tetrahedron has volume 611,520. Heron's classical formula – $\sqrt{s(s-a)(s-b)(s-c)}$, where s is the semiperimeter of a triangle – shows that the faces have area 13,650.

No solutions were submitted to part (b) and it remains open.