MAXIMIZING THE SURFACE AREA OF AN N-DIMENSIONAL UNIT SPHERE

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Using the Dirichlet integral in n-dimensional Euclidean space, one can show that the volume of an n-dimensional sphere with radius r is given by

$$V_n(r) = \frac{\pi^{\frac{n}{2}} r^n}{\Gamma(\frac{n}{2}+1)} \ ,$$

where n is a positive integer. Of course, 'volume' V_1 is the length of the interval $[-r, r]$ and 'volume' V_2 is the area of the circle with radius r. So, $V_1(r) = 2r$ and $V_2(r) = \pi r^2$.

The surface area of an n-dimensional sphere with radius r is given by

$$\begin{aligned} S_n(r) &= V_n'(r) \\ &= \frac{n\pi^{\frac{n}{2}} r^{n-1}}{\frac{n}{2}\Gamma(\frac{n}{2})} \\ &= \frac{2\pi^{\frac{n}{2}} r^{n-1}}{\Gamma(\frac{n}{2})} \ . \end{aligned}$$

In [1], it was shown that $V_n(1)$ has a maximum value when $n = 5$. The purpose of this paper is to show that $S_n = S_n(1)$ attains a maximum value for $n = 7$. This will be accomplished by showing that

(a) $S_7 > S_6 > S_5 > S_4 > S_3 > S_2 > S_1$ and

(b) $\{S_n\}_{n=7}^{\infty}$ is a decreasing sequence.

Using Table 1, it is easy to verify (a).

n	S_n	Approximate value of S_n
1	2	2
2	2π	6.283
3	4π	12.566
4	$2\pi^2$	19.739
5	$8\pi^2/3$	26.319
6	π^3	31.006
7	$16\pi^3/15$	33.073
8	$\pi^4/3$	32.470
9	$32\pi^4/105$	29.687

Table 1

Before proofing (b), notice that

$$S_{n+1} = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$$

$$= \frac{\sqrt{\pi}\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} S_n \ . \tag{1}$$

To prove (b), it will be shown by mathematical induction that

(i) $S_{2n+1} > S_{2n+2}$ for integers $n \geq 3$ and

(ii) $S_{2n} > S_{2n+1}$ for integers $n \geq 4$.

From Table 1, inequality (i) is valid for $n = 3$. For the induction hypothesis, assuming that $S_{2k+1} > S_{2k+2}$ for some integer $k \geq 3$ is equivalent to assuming that

$$\Gamma(k+1) > \sqrt{\pi}\, \Gamma\left(\frac{2k+1}{2}\right) \ .$$

Now, from (1)

$$\frac{S_{2k+3}}{S_{2k+4}} = \frac{\Gamma(k+2)}{\sqrt{\pi}\ \Gamma(\frac{2k+3}{2})}$$

$$= \frac{(k+1)\Gamma(k+1)}{(\frac{2k+1}{2})\sqrt{\pi}\ \Gamma(\frac{2k+1}{2})}$$

$$> \frac{2k+2}{2k+1} > 1\ .$$

So, (i) has been verified by mathematical induction.

Again, from Table 1, (ii) holds for $n = 4$. Assume that $S_{2k} > S_{2k+1}$ for some $k \geq 4$. From (1), this assumption is equivalent to

$$\Gamma\left(\frac{2k+1}{2}\right) > \sqrt{\pi}\ \Gamma(k)$$

for some $k \geq 4$. Hence,

$$\frac{S_{2k+2}}{S_{2k+3}} = \frac{\Gamma(\frac{2k+3}{2})}{\sqrt{\pi}\ \Gamma(k+1)}$$

$$= \frac{(\frac{2k+1}{2})\Gamma(\frac{2k+1}{2})}{\sqrt{\pi}k\Gamma(k)}$$

$$> \frac{2k+1}{2k} > 1\ .$$

Hence, inequality (ii) holds for all integers $n \geq 4$. This completes the proof of (b).

Reference

1. R. Salgia, "Volume of an n-Dimensional Unit Sphere," *Pi Mu Epsilon Journal*, 2 (Spring 1983), 496–501.