A NOTE ON LOWER NEAR FRATTINI SUBGROUPS

Kandasamy Muthuvel

University of Wisconsin-Oshkosh

Theorem 1 in [1] reads, "Let H be a normal subgroup of a group G such that the order of H is prime. Let $\lambda(G)$ denote the set of all non-near generators of G. Then $\lambda(G) \cap H = \{1\}$ if and only if G nearly splits over H." The purpose of this note is to show that Theorem 1 in [1] may be improved as follows: If H is a normal subgroup of a group G and H is of prime order, more generally, if H is a finite cyclic normal subgroup of a group G, then $H \subseteq \lambda(G)$ and G does not nearly split over H. We also prove that if the condition "H is finite" is replaced by "H is infinite" in the above statement, then $\lambda(G) \cap H = \{1\}$ if and only if G nearly splits over H.

We first recall some definitions (see [1] or [2]).

Definition 1. An element g of a group G is a non-near generator of G if $S \subseteq G$ and $|G :< g, S > |$ is finite implies $|G :< S > |$ is finite. The set of all non-near generators of G, denoted by $\lambda(G)$, is called the lower near Frattini subgroup of G.

Definition 2. Let H be a normal subgroup of a group G. We say that G nearly splits over H if there exists a subgroup K of G such that $|G : K|$ is infinite, $|G : HK|$ is finite and

$$\bigcap_{g \in G} g^{-1}(H \cap K)g = \{1\}.$$

Lemma 1. If S is a subset of a group G and x is an element of G such that $|G :< S > |$ is infinite and $< x >$ is a finite normal subgroup of G, then $|G :< h, S > |$ is infinite for every $h \in < x >$.

Proof. Let $g \in G$ and $|x| = n$. Then

$$g < x, S >= g < x >< S >= \bigcup_{1 \leq i \leq n} gx^i < S > .$$

That is, any left coset of $< x, S >$ in G is a finite union of left cosets of $< S >$ in G and hence if $|G :< x, S > |$ is finite, then G is a finite union of left cosets of $< S >$ in G (i.e.

$|G :< S > |$ is finite). Thus $|G :< x, S > |$ is infinite. Now let $h \in< x >$. Then $h = x^m$ for some positive integer m. Since $< x, S >\supseteq< x^m, S >$ and $|G :< x, S > |$ is infinite, $|G :< x^m, S > |$ is infinite. This completes the proof.

Proposition 1. If H is a finite cyclic normal subgroup of a group G, then $H \subseteq \lambda(G)$ and G does not nearly split over H.

Proof. If $H \not\subseteq \lambda(G)$, then there exists an h in H such that h is not in $\lambda(G)$. $h \notin \lambda(G)$ implies that there exists a subset S of G such that $|G :< S > |$ is infinite and $|G :< h, S > |$ is finite. This is impossible by Lemma 1. Let $H =< x >$. If G nearly splits over H, then there exists a subgroup K of G such that $|G : HK| = |G :< x, K > |$ is finite and $|G : K|$ is infinite. Again by Lemma 1, this is impossible.

Remark 1. It is interesting to note that the proof of part 1 of Proposition 1 works for any subgroup H of G such that every cyclic subgroup of H is a finite normal subgroup of G.

Lemma 2. Let $< x >$ be an infinite cyclic normal subgroup of a group G. If S is a subset of G such that $|G :< S > |$ is infinite and $|G :< x, S > |$ is finite, then $< x > \cap < S >= \{1\}$.

Proof. Suppose $< x > \cap < S >\neq \{1\}$. Then there exists a smallest positive integer n such that $x^n \in< x > \cap < S >$. Hence

$$< x >< S >= \bigcup_{1\leq i\leq n} x^i < S > .$$

This together with the hypothesis $|G :< x, S > |$ is finite imply that G is a finite union of left cosets of $< S >$ in G, which contradicts that $|G :< S > |$ is infinite. Thus $< x > \cap < S >= \{1\}$.

Theorem 1. Let H be an infinite cyclic normal subgroup of a group G. Then $\lambda(G) \cap H = \{1\}$ if and only if G nearly splits over H.

Proof. Let $H =< x >$. If $\lambda(G) \cap H = \{1\}$, then there exists a subset S of G such that $|G :< S > |$ is infinite and $|G :< x, S > |$ is finite. By Lemma 2 and by using the fact that $< x, S >=< x >< S >$, it can be easily seen that G nearly splits over H. Conversely, if G nearly splits over H, then there exists a subgroup K of G such that $|G : K|$ is infinite and $|G : HK|$ is finite. $|G : HK| = |G :< x, K > |$ is finite implies that for $i \neq 0, |G :< x^i, K > |$

is finite, because

$$< x, K > = < x > K \subseteq \bigcup_{-|i| \leq j \leq |i|} x^j < x^i, K > \ .$$

Hence for $i \neq 0, x^i \notin \lambda(G)$. Thus $\lambda(G) \cap H = \{1\}$.

Remark 2. Theorem 1 in [1] reads: "Let H be a normal subgroup of a group G such that the order of H is prime. Then $\lambda(G) \cap H = \{1\}$ if and only if G nearly splits over H. In other words $\lambda(G) \cap H \neq \{1\}$ if and only if G does not nearly split over H." The proof of Theorem 1 in [1] is correct, but it follows from Proposition 1 that under the hypotheses of Theorem 1 in [1], we get a much stronger conclusion, namely $H \subseteq \lambda(G)$ and G does not nearly split over H.

References

1. M. K. Azarian, "On Lower Near Frattini Subgroups and Nearly Splitting Groups," *Missouri J. Math. Sci.* 2 (1990), 18-25.

2. J. B. Riles, "The Near Frattini Subgroups of Infinite Groups, " *J. of Algebra* 12(1969), 155-171.