NOTE ON THE FIRST FUNDAMENTAL THEOREM FOR RIEMANN-STIELTJES INTEGRALS

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A Riemann-Stieltjes integral is of the form $\int_a^b f d\alpha$. The functions f and α are called the integrand and integrator, respectively. If the integral exists, we say $f \in R(\alpha)$ on $[a, b]$. We will always assume the integrator is of bounded variation (a sufficient condition for this is that α be monotonic, although this is not necessary). This insures that if $f \in R(\alpha)$ on $[a, b]$ then $f \in R(\alpha)$ on $[a, x]$ also, for all $x \in [a, b]$. Thus the function $F(x) = \int_a^x f d\alpha$ is well-defined on $[a, b]$. The special case of the Riemann integral occurs when $\alpha(x) = x$.

The first fundamental theorem of calculus for the Riemann-Stieltjes integral applies when the integrator is monotonic on $[a, b]$. In this case,

$$\frac{d}{dx} \int_a^x f d\alpha = f(x)\alpha'(x)$$

for each $x \in (a, b)$ at which f is continuous and α' exists. Thus for Riemann integrals, the only requirement is the continuity of f at x.

The requirement that α be monotonic can be dropped if α' is continuous on $[a, b]$. In this case, the Riemann-Stieltjes integral reduces to a Riemann integral: $\int_a^x f d\alpha = \int_a^x f\alpha'$. Thus the only requirement for the differentiability of this integral at x is the continuity of f at x. For proofs of these results, see [1].

The fundamental theorem can be easily extended to include all integrators α of bounded variation on the interval $[a, b]$ for which

(1) α' exists on (a, b), and

(2) for each $x \in (a, b)$, there is a neighborhood of x on which α' is bounded.

For instance, the fundamental theorem applies, even at $x = 0$, with the integrator

$$\alpha(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

To prove the extended version, assume for some neighborhood of x that $\alpha = \beta + \gamma$ where β and γ are monotonic and differentiable at x. The extended version now follows from the monotonic version, using the linearity of integration with respect to a sum of integrators.

To complete the proof, given $x \in (a, b)$, choose a neighborhood B of x and constant M such that $M \geq |\alpha'(t)|$ for all $t \in B$. On B, define $\beta(t) = Mt$ and $\gamma = \alpha - \beta$. Now γ is decreasing on B since if $t_1 < t_2$ then, by the Mean Value Theorem [1],

$$\begin{aligned} \gamma(t_1) - \gamma(t_2) &= \alpha'(c)(t_1 - t_2) - M(t_1 - t_2) \\ &= (\alpha'(c) - M)(t_1 - t_2) \geq 0 \ , \end{aligned}$$

for some $c \in (t_1, t_2)$.

<u>*Reference*</u>

1. T. M. Apostol, *Mathematical Analysis*, 2nd edition, Addison-Wesley, 1975.