APPLICATION OF TRANSLATION OF SETS

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In this paper, we show that if a set A of reals contains a translated copy of every finite set then for any (additive) proper subgroup F of R, $A \setminus F$ contains a translated copy of every finite set. This implies that the real line is not the finite union of proper subgroups of R. It is interesting to note that no group is the union of two proper subgroups but there is a group, namely $\{1, 3, 5, 7\}$ under addition modulo 8, which is the union of three proper subgroups. Existence of small sets of reals (in the sense of category and measure) containing a translated copy of every countable set is proved in [3].

Throughout this paper, R denotes the set of all real numbers, N is the set of all positive integers and R^* denotes the set of all nonzero reals.

Proposition 1. If F is a proper subgroup of R, then $|R : F|$, the index of F in R is infinite.

Proof. Suppose $|R : F|$ is finite. Then R is a finite union of left cosets $F + x_i$, $1 \leq i \leq m$. Since $\{nx_i : n \in N\}$ is infinite for each i, for infinitely many n, nx_i belongs to the same coset, say $F + x_j$. Hence there exists a smallest positive integer Y_i such that $Y_i x_i \in F$, because $n_1 x_i$ and $n_2 x_i \in F + x_j$ for some n_1, n_2 imply that $(n_1 - n_2)x_i \in F$. Let ℓ be the least common multiple of Y_i for $1 \leq i \leq m$. Then $\ell x_i \in F$ for every $i \leq m$. For any $r \in R$, $r = f + x_i$ for some $f \in F$ and some $i \leq m$. Hence $\ell r \in F$ for all $r \in R$, and consequently $\ell\left(\frac{r}{\ell}\right) = r \in F$ for all $r \in R$, which contradicts that F is a proper subgroup of R.

Corollary 1. R is not the direct sum of a cyclic subgroup and a proper subgroup of R.

Proof. If $R =< g > +F$, then $\frac{g}{2} = ng + f$ for some $n \in Z$ and $f \in F$. Then $(2n - 1)g \in F$ and consequently R is a finite union of left cosets of F in G, contradicting Proposition 1.

Remark 1. It can be easily seen from the proof of Proposition 1 that if G is any additive subgroup of R such that $\frac{g}{n} \in G$ for all $g \in G$ and $n \in N$ or G is the multiplicative subgroup of R^* such that $g^{\frac{1}{n}} \in G$ for all $g \in G$ and $n \in N$, then G contains no proper subgroup of finite index. For example, the set of all rationals under addition or the set of all positive real numbers under multiplication contains no proper subgroup of finite index.

Proposition 2. If a subset of A of R contains a translated copy of every finite set, then for any proper subgroup F of R, $A \setminus F$ contains a translated copy of every finite set.

Proof. Let H be a finite set of n elements. Since $|R : F|$ is infinite, $R \supseteq F + r_i$ for every $i \in N$, where $r_i - r_j \notin F$ whenever $i \neq j$ and $i, j \in N$. Since $H + \{r_i : 1 \leq i \leq n+1\}$ is finite, $H + \{r_i : 1 \leq i \leq n+1\} + r \subseteq A$ for some $r \in R$. It suffices to prove that $H + r_i + r \cap F = \emptyset$ for some $i \leq n+1$. Suppose $H + r_i + r \cap F \neq \emptyset$ for every $i \leq n+1$. Then for each $i \leq n+1$, there exists $h_i \in H$ such that $h_i + r_i + r \in F$. Since $|H| = n$ and there are $n+1$ values for i, $h_j = h_\ell$ for some $j \neq \ell$ and $1 \leq j, \ell \leq n+1$. Hence

$$h_j + r_j + r - (h_\ell + r_\ell + r) = r_j - r_\ell \in F \ ,$$

which contradicts the fact $r_j - r_\ell \notin F$.

Corollary 2. R is not the finite union of proper subgroups of R.

Proof. Suppose

$$R = \bigcup F_i \ ,$$

where F_i is a proper subgroup of R and $1 \leq i \leq n$. Then by Proposition 2, $R \setminus F_1$ contains a translated copy of every finite set. By continuing in this fashion,

$$R \setminus \bigcup_{1 \leq i \leq n} F_i = \emptyset$$

contains a translated copy of every finite set. This is impossible.

Definition. The algebraic difference of a set A of reals, denoted by $D(A)$, is defined to be $\{x - y : x, y \in A\}$.

A classical result of Piccard [4] states that if A is a Baire set and is of second category, then $D(A) = A - A$ contains an interval.

The following lemma (stated without proof in [2]) is slightly stronger than Piccard's theorem.

Lemma 1. If Q is a set of the first Baire category and $a < b$, then

$$D((a, b) \setminus Q) = D(a, b) = (a - b, b - a).$$

First we shall prove a lemma, which is slightly stronger than the above lemma, to prove a proposition.

Lemma 2. Let (a,b) be an interval and x a fixed element in (a,b). If a set A contains a translated copy of $\{x,y\}$ for every $y \in (a,b)$, then $D(A) \supseteq (a-x, b-x)$.

Proof. If not, find an element i in $(a-x, b-x) \setminus D(A)$. Then $\{x, x+i\} + r \subseteq A$ for some r in R, and consequently $i \in A - A = D(A)$, which contradicts the choice of i.

Proposition 3. Let $a < b$. If Q is a set of the first Baire category, or Q is a subgroup of R of uncountable index, then $D((a,b) \setminus Q) = (a-b, b-a)$.

Proof. Let x be a fixed element in (a,b). Suppose Q is a set of the first Baire category. Then for every y in (a,b), $\{x,y\} + r \subseteq (a,b)$ for all $r \in (0, b - \max\{x,y\}) = I$ (say). If $\{x,y\} + r \cap Q \neq \emptyset$ for every $r \in I$, then there exists a second category subset J of I such that for all $r \in J$, $x + r \in Q$ or $y + r \in Q$ for all $r \in J$. This contradicts that Q is of first category. Thus $(a,b) \setminus Q$ satisfies the hypothesis of Lemma 2. Now, suppose Q is a subgroup of R of uncountable index. If $\{x,y\} + r \cap Q \neq \emptyset$ for all $r \in I$, then $I \subseteq (Q-x) \cup (Q-y)$. By using the fact that any open interval of length $|I|$ is a translated copy of I, it can be easily seen that R is contained in a countable union of sets of the form $Q - x$, which contradicts that $|R : Q|$ is uncountable. Thus $(a,b) \setminus Q$ contains a translated copy of $\{x,y\}$ for every $y \in (a,b)$.

In both cases, by Lemma 2, $D((a,b) \setminus Q) \supseteq (a-x, b-x)$ for every $x \in (a,b)$ and consequently $D((a,b) \setminus Q) \supseteq (a-b, b-a)$. Trivially $D((a,b) \setminus Q) \subseteq (a-b, b-a)$. This completes the proof.

References

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