A COUNTER EXAMPLE IN GROUP THEORY

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In a first course on basic group theory, one of the standard problems is to show that if $G = \{x_1, x_2, \cdots, x_n\}$ is an abelian group and n is odd then the product $x_1 x_2 \cdots x_n = e$, where e is the identity element G. In this short note, we give a counter example to show that the above result is not true if we drop the 'abelianness' of the group. In looking for an example, we do not need to consider a group of order 3, 5, 7, 11, 13, 17, 19, because these are primes and any group of prime order is cyclic and hence abelian. Also $n = 9$ does not work, because it is the square of a prime and hence the group is abelian. Also by fairly standard arguments, one can see that group of order 15 is abelian. Therefore the first possible candidate for a counter example is a group of order 21. Apart from the cyclic group of order 21, there is a unique non-abelian group of order 21 (refer to p. 112, problem 11(b) of [1]). We show that this unique non-abelian group G of order 21 works as a counter example. As a matter of fact, we find an arrangement $x_1, x_2, \cdots, x_{20}$ of non-identity elements of G such that the product $x_1 x_2 \cdots x_{20}$ is non-identity. Let $a, b \epsilon G$ such that order of a and b be 3 and 7 respectively and e be the identity element of G. Let

$$G = \{e, a, a^2, b^i, ab^i, a^2b^i : 1 \leq i \leq 6\}.$$

Since $a^{-1} = a^2$, and $\{e, b^i, 1 \leq i \leq 6\}$ is a normal subgroup of G (by Sylow Theorem), and since $ab \neq ba$, there exists an $i, 2 \leq i \leq 6$, such that $aba^2 = b^i$.

Now we let,

$x_1 = ab^2,$	$x_5 = ab^4,$	$x_9 = ab^6,$	$x_{13} = a,$	$x_{17} = b^3,$
$x_2 = a^2b^2,$	$x_6 = a^2b^4,$	$x_{10} = a^2b^6,$	$x_{14} = a^2b,$	$x_{18} = b^4,$
$x_3 = ab^3,$	$x_7 = ab^5,$	$x_{11} = a^2,$	$x_{15} = b,$	$x_{19} = b^5,$
$x_4 = a^2b^3,$	$x_8 = a^2b^5,$	$x_{12} = ab,$	$x_{16} = b^2,$	$x_{20} = b^6.$

Since $aba^2 = b^i$, the product

$$\begin{aligned} x_1 x_2 \cdots x_{10} &= b^{2i+2} b^{3i+3} \cdots b^{6i+6} \\ &= b^{i(2+3+\cdots+6)+(2+3+\cdots+6)} \\ &= b^{20i+20} = b^{20(i+1)}. \end{aligned}$$

Also since $a^3 = e$ and $b^7 = e$, the product $x_{11}x_{12}\cdots x_{20} = b^2$. Hence the product $x_1 x_2 \cdots x_{20} = b^{20(i+1)+2}$. Since 7 does not divide $20(i+1)+2$ for $2 \le i \le 6$, the product $x_1 x_2 \cdots x_{20} \neq e$.

Remark. By Proposition 6.1, p. 97 [2], it can be seen that the value of i in the above argument can only be either 2 or 4 but not both. But this is not relevant in the above.

References

1. I. N. Herstein, *Topics in Algebra*, John Wiley and Sons, Inc., 1975.
2. T. W. Hungerford, *Algebra*, Springer-Verlag, New York, 1974.