DIFFERENTIATING INDEFINITE INTEGRALS WITH RESPECT TO A PARAMETER

Joseph Wiener and Donald P. Skow

The University of Texas-Pan American

In teaching calculus, it is important to demonstrate the close relationship between the operations of differentiation and integration. After students finish differential calculus, they are usually introduced to the concept of integration as an operation inverse to differentiation. However, after this they see limited use of the derivative in integral calculus. True, in advanced calculus courses differentiation with respect to a parameter is used to evaluate some complicated improper integrals [1]. The purpose of this note is to show that the powerful method of differentiation with respect to a parameter can be applied to obtain closed forms of certain classes of indefinite integrals.

This approach is based upon Leibniz's formula for the nth derivative of a product of n times differentiable functions $u(x)$ and $v(x)$, namely

$$\text{(L)} \qquad (uv)^{(n)} = \sum_{i=0}^{n} \binom{n}{i} u^{(i)} v^{(n-i)}.$$

The method enables us to integrate products of a power function by an exponential function or sine(cosine) and therefore may be considered as a nice supplement to integration by parts. The ideas are not new but several examples will show the value of this method.

Example 1. The calculus textbooks and handbooks of mathematics list the indefinite integral

$$\int x^n e^{ax} dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} dx + C,$$

the degree of which is repeatedly reduced by integration by parts. Let p be a parameter and the constant of integration C be a function of p. We consider the integral

$$\int e^{px}dx = p^{-1}e^{px} + C, \quad p \neq 0. \tag{1}$$

Differentiating both sides of (1) n times with respect to p, by means of formula (L) with $u = p^{-1}$ and $v = e^{px}$ yields the closed form expression

$$\int x^n e^{px}dx = \sum_{i=0}^{n} \frac{n!(-1)^i(i!)(p^{-1-i})x^{n-i}e^{px}}{(i!)(n-i)!}$$

$$= n!\ p^{-1}e^{px} \sum_{i=0}^{n} \frac{(-1)^i p^{-i} x^{n-i}}{(n-i)!} + C.$$

Example 2. Consider the integral

$$\int x^{p-1}dx = p^{-1}x^p + C, \quad p \neq 0. \tag{2}$$

Differentiating both sides of (2) n times with respect to p, with $u = p^{-1}$ and $v = x^p$, we get

$$\int x^{p-1}(\ln x)^n dx = \sum_{i=0}^{n} \frac{(n!)(-1)^i(i!)(p^{-i-i})(x^p)(\ln x)^{n-i}}{(i!)(n-i)!}$$

$$= n!\ p^{-1}x^p \sum_{i=0}^{n} \frac{(-1)^i(p^{-i})(\ln x)^{n-i}}{(n-i)!} + C.$$

Example 3. Consider

$$\int \cos(px)dx = p^{-1}\sin(px) + C, \quad p \neq 0 \tag{3}$$

and differentiate (3) n times with respect to p, taking $u = \sin(px)$ and $v = p^{-1}$. This gives

$$\int x^n \cos(px + \frac{n\pi}{2})dx$$

$$= \sum_{i=0}^{n} \frac{n!(-1)^{n-i}(n-i)!(p^{-1-n+i})x^i \sin(px + \frac{i\pi}{2})}{(i!)(n-i)!}$$

$$= n!\ p^{-1-n} \sum_{i=0}^{n} \frac{(-1)^{n-i}(px)^i \sin(px + \frac{i\pi}{2})}{i!} + C.$$

Here we used the formulas $(\sin x)^{(n)} = x^n \sin(x + \frac{n\pi}{2})$ and $(\cos x)^{(n)} = x^n \cos(x + \frac{n\pi}{2})$. If n is even, then $n = 2m$ and $\cos(px + \frac{n\pi}{2}) = (-1)^m \cos(px)$. Substituting and simplifying, the integral becomes

$$\int x^{2m} \cos(px)dx = (-1)^m (2m!)p^{-2m-1} \sum_{i=0}^{2m} \frac{(-1)^i (px)^i \sin(px + \frac{i\pi}{2})}{i!} + C.$$

If n is odd, then $n = 2m+1$ and $\cos(px + \frac{n\pi}{2}) = (-1)^{m+1} \sin(px)$. Substituting and simplifying, the integral becomes

$$\int x^{2m+1} \sin(px)dx =$$

$$(-1)^m (2m+1)!\ p^{-2m-2} \sum_{i=0}^{2m+1} \frac{(-1)^i (px)^i \sin(px + \frac{i\pi}{2})}{i!} + C.$$

The reader might try, as an exercise, the integrals:

$$\int x^{2m} \sin(px)dx =$$

(A)

$$(-1)^{m+1}(2m!)\ p^{-2m-1} \sum_{i=0}^{2m} \frac{(-1)^i (px)^i \cos(px + \frac{i\pi}{2})}{i!} + C$$

and

$$\text{(B)} \qquad \begin{aligned} &\int x^{2m+1}\cos(px)dx = \\ &\qquad (-1)^m(2m+1)!\; p^{-2m-2}\sum_{i=0}^{2m+1}\frac{(-1)^i(px)^i\cos(px+\frac{i\pi}{2})}{i!}+C. \end{aligned}$$

<u>Example 4</u>. Consider the integral

$$\text{(4)} \qquad \int\frac{dx}{(x+q)(x+p)} = (p-q)^{-1}\ln|x+q|-(p-q)^{-1}\ln|x+p|+C, \quad p\neq q.$$

Differentiating both sides of (4) $n-1$ times with respect to p, with $u=(p-q)^{-1}$ and $v=\ln|x+p|$, (4) gives, when simplified

$$\text{(5)} \qquad \begin{aligned} &\int\frac{dx}{(x+q)(x+p)^n} = \\ &\frac{\ln|x+q|-\ln|x+p|}{(p-q)^n}+\sum_{i=0}^{n-2}\frac{1}{(n-1-i)(p-q)^{1+i}(x+p)^{n-1-i}}+C, \quad p\neq q. \end{aligned}$$

Now differentiating formula (5) $m-1$ times with respect to q, with $u=(p-q)^{-n}$ and

$v = \ln|x+q|$, (5) yields

$$\int \frac{dx}{(x+q)^m(x+p)^n} =$$

$$\sum_{k=0}^{m-2} \frac{(-1)^{1-k}(n+k-1)!}{k!(m-k-1)(n-1)!(p-q)^{n+k}(x+q)^{m-1-k}}$$

$$+ \frac{(-1)^{m-1}(n+m-2)!}{(m-1)!(n-1)!(p-q)^{n+m-1}} \left[\ln|x+q| - \ln|x+p|\right]$$

$$+ \frac{(-1)^{m-1}}{(m-1)!} \sum_{i=0}^{n-2} \frac{(i+m-1)!}{(n-1-i)(x+p)^{n-1-i} i!\ (p-q)^{i+m}} + C.$$

References

1. J. Wiener, "An Analytical Approach to a Trigonometric Integral," *Missouri J. Math. Sci.,* Spring 1990, Vol. 2. No. 2, 75-77.

2. C. H. Edwards, Jr. and D. E. Penney, *Calculus and Analytic Geometry*, Third Ed., Prentice Hall, 1990.