SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the editor.

17. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Let $ABCD$ be an isosceles tetrahedron. (An isosceles tetrahedron is a tetrahedron whose opposite edges are equal.) Denote the dihedral angle at edge AB by $\angle AB$. Prove that

$$\frac{AB}{\sin \angle AB} = \frac{AC}{\sin \angle AC} = \frac{AD}{\sin \angle AD} \ .$$

Solution by the proposer.

Let AH be the altitude from A to face BCD and let AE be the altitude from A to BC in face ABC. Then triangle AHE is a right triangle with $\angle AEH = \angle BC$. Hence $\sin \angle BC = AH/AE$. But $AH = 3V/K$, where V is the volume of the tetrahedron and K is the area of any face; and $AE = 2K/BC$. Therefore $\sin \angle BC = 3V(BC)/2K^2$ and hence

$$\frac{\sin \angle BC}{BC}$$

is constant. Since BC was arbitrary, this is true for all six edges of the tetrahedron.

18. *Proposed by Jayanthi Ganapathy, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.*

Let f be differentiable on an interval of the form (M, ∞). Suppose

$$\lim_{x\to\infty} \big(f(x) + xf'(x)\big) = \alpha \ ,$$

where α is finite. Prove

$$\lim_{x\to\infty} f(x) \qquad \text{and} \qquad \lim_{x\to\infty} f'(x)$$

exist and evaluate these limits.

Solution I by Robert Doucette (student), University of Iowa, Iowa City, Iowa.

Let $g : (M, \infty) \to R$ be defined by $g(x) = xf(x)$. The hypothesis becomes

$$\lim_{x\to\infty} g'(x) = \alpha \ .$$

Let $\epsilon > 0$. We may choose $x_0 > \max\{M, 0\}$ such that $x > x_0$ implies

$$|g'(x) - \alpha| < \frac{\epsilon}{4} \ .$$

Next choose $x_1 > x_0$ such that

$$x_1\big(\frac{\epsilon}{4}\big) > x_0\big(|\alpha| + \frac{\epsilon}{4}\big) \quad \text{and} \quad |g(x_0)| \ .$$

Suppose $x > x_1$. By the Mean Value Theorem, there is a $c > x_0$ such that

$$g(x) - g(x_0) = g'(c)(x - x_0) \ .$$

With this

$$\begin{aligned} \Big|\frac{g(x) - g(x_0)}{x} - \alpha\Big| &\le |g'(c) - \alpha| + |g'(c)|\frac{x_0}{x} \\ &< \frac{\epsilon}{4} + \big(|\alpha| + \frac{\epsilon}{4}\big)\frac{x_0}{x_1} \\ &< \frac{\epsilon}{2} \ . \end{aligned}$$

Therefore

$$\begin{aligned} |f(x) - \alpha| = \Big|\frac{g(x)}{x} - \alpha\Big| &\le \Big|\frac{g(x) - g(x_0)}{x} - \alpha\Big| + \frac{|g(x_0)|}{x} \\ &< \frac{\epsilon}{2} + \frac{|g(x_0)|}{x_1} \\ &< \epsilon \ . \end{aligned}$$

This shows that

$$\lim_{x\to\infty} f(x) = \alpha \ .$$

Also

$$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} \frac{g'(x) - f(x)}{x} = 0 \ .$$

Solution II by Don Redmond, Southern Illinois University, Carbondale, Illinois.

We use two lemmas that can be found in G.H. Hardy, *A Course in Pure Mathematics*, 10th ed., p. 280.

Lemma 1. Suppose that

$$\lim_{x\to+\infty} \phi'(x) = a \ ,$$

where a is finite. If $a \neq 0$, then

$$\lim_{x\to+\infty} \frac{\phi(x)}{ax} = 1$$

and if $a = 0$, then

$$\lim_{x\to+\infty} \frac{\phi(x)}{x} = 0 \ .$$

Lemma 2. If

$$\lim_{x\to+\infty} \psi(x) = a \ ,$$

where a is finite, then

$$\lim_{x\to+\infty} \psi'(x) = 0 \ .$$

In the problem note that

$$f(x) + xf'(x) = (xf(x))' \ .$$

Thus the statement of the problem is that

$$\lim_{x\to+\infty} (xf(x))' = \alpha \ ,$$

where α is finite. By Lemma 1 we see that, if $\alpha \neq 0$

$$\lim_{x\to+\infty} \frac{xf(x)}{\alpha x} = 1 \ ,$$

that is,

$$\lim_{x\to+\infty} f(x) = \alpha \ ;$$

and if $\alpha = 0$, then

$$\lim_{x \to +\infty} \frac{xf(x)}{x} = 0 \ ,$$

that is,

$$\lim_{x \to +\infty} f(x) = 0 \ .$$

Thus we see that

$$\lim_{x \to +\infty} f(x) = \alpha \ ,$$

where α is finite, and so, by Lemma 2, we see that

$$\lim_{x \to +\infty} f'(x) = 0 \ .$$

Solution III by the proposer.

Since

$$f(x) = f(x) \cdot \frac{x}{x}$$

we see by L'Hôpital's Rule that

$$\begin{aligned} \lim_{x \to \infty} f(x) &= \lim_{x \to \infty} \frac{xf(x)}{x} \\ &= \lim_{x \to \infty} \frac{f(x) + xf'(x)}{1} \\ &= \alpha \ . \end{aligned}$$

Hence

$$\lim_{x \to \infty} f(x) = \alpha$$

and

$$\lim_{x \to \infty} xf'(x) = 0$$

so that

$$\lim_{x \to \infty} f'(x) = \lim_{x \to \infty} xf'(x) \cdot \frac{1}{x} = 0 \ .$$

If the condition

$$\lim_{x\to\infty}\big(f(x)+xf'(x)\big)=\alpha$$

is changed to

$$\lim_{x\to\infty}\big(kf(x)+f'(x)\big)=\alpha \quad , \ k>0 \ ,$$

the conclusion about the existence of limits still holds. Follow the same argument more or less except use the fact that

$$f(x)=f(x)\cdot\frac{e^{kx}}{e^{kx}}$$

and apply L'Hôpital's Rule.

19. *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Prove that triangle ABC is isosceles if and only if

$$\tan(A-B) \ + \ \tan(B-C) \ = \ \tan(A-C) \ .$$

Solution by Stanley Rabinowitz, Westford, Massachusetts.

With the given information, the following statements are equivalent:

$$\begin{aligned}\tan\big[(A-B)+(B-C)\big] &= \frac{\tan(A-B)+\tan(B-C)}{1-\tan(A-B)\tan(B-C)}\\ \tan(A-C) &= \frac{\tan(A-C)}{1-\tan(A-B)\tan(B-C)}\end{aligned}$$

$$\tan(A-C)-\tan(A-C)\tan(A-B)\tan(B-C)=\tan(A-C)$$

$$\tan(A-C)\tan(A-B)\tan(B-C)=0$$

$$\tan(A-C)=0 \ \text{ or } \ \tan(A-B)=0 \ \text{ or } \ \tan(B-C)=0 \ .$$

Hence, since A, B, C are angles of a triangle, so that $-\pi < A - B < \pi$, etc., we must have $A - C = 0$ or $A - B = 0$ or $B - C = 0$, i.e. $\triangle ABC$ is isosceles.

Note that we did not actually need the fact that $A + B + C = \pi$.

Also solved by Donald Fuller, Gainesville College, Gainesville, Georgia; Jayanthi Ganapathy, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Donald Skow, University of Texas-Pan American, Edinburg, Texas; and the proposer.

20. *Proposed by Troy Hicks, University of Missouri-Rolla, Rolla, Missouri.*

The useful lemma given below is part of the folklore of topology. A non-elementary proof was given in [1] and the lemma was used to prove the following theorem. A Tychonoff space X is separable and metrizable if and only if every compatible uniformity on X contains a compatible totally bounded uniformity. Then applications of this result were given to group actions, compactifications, and proximity classes of uniformities. It seems appropriate to have an elementary proof in the literature. Supply one.

Lemma. Let $(X, \mathcal{U})$ be a uniform space. Then for any closed subset A of X and any point $z \in X - A$, there is a uniformily continuous function $f : (X, \mathcal{U}) \to [0, 1]$ such that $f(z) = 0$ and $f(A) = 1$.

Reference

1. P. L. Sharma and T. L. Hicks, "Uniformities on Separable Metrizable Spaces," *Math. Japonica*, 25, No. 6 (1980), 677–680.

Solution by the proposer.

The following two results can be found in standard texts. They will be used without proof.

(a) Every uniformity can be derived from the family $G(\mathcal{U})$ of its uniformily continuous pseudo-metrices; that is

$$\{V_{\rho,r} : \rho \in G(\mathcal{U}) \text{ and } r > 0\}$$

is a base for $\mathcal{U}$ where

$$V_{\rho,r} = \{(x,y) : \rho(x,y) < r\} \ .$$

(b) $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is uniformily continuous if and only if for every $d \in G(\mathcal{V})$ and each $\epsilon > 0$, there exists $\rho \in G(\mathcal{U})$ and $\delta > 0$ such that, if $\rho(x,y) < \delta$, then $d(f(x), f(y)) < \epsilon$.

Proof. From (a), there exists $\rho \in G(\mathcal{U})$ and $r > 0$ such that

$$V_{\rho,r}[z] = \{y : \rho(z,y) < r\} \subset X - A \ .$$

Define f by

$$f(y) = \ \min \ \{1, \rho(z,y)/r\} \ .$$

Observe that $f(z) = 0$ and $f(A) = 1$. We show that f is uniformily continuous. Given $0 < \epsilon < 1$, we may assume $0 < r < \epsilon$. From (b), it suffices to prove that $\rho(x,y) < r\epsilon = \delta$ implies $|f(x) - f(y)| < \epsilon$.

Case 1. $x, y, \in X - V_{\rho,r}[z]$. Then $f(x) = f(y) = 1$.

Case 2. $x, y, \in V_{\rho,r}[z]$. Then

$$|f(x) - f(y)| = |\rho(z,x)/r - \rho(z,y)/r| = \rho(x,y)/r < r\epsilon/r = \epsilon \ .$$

Case 3. $x \in V_{\rho,r}[z]$ and $y \notin V_{\rho,r}[z]$. Then

$$r \leq \rho(y,z) \leq \rho(y,x) + \rho(x,z)$$

or

$$r - \rho(x,z) \leq \rho(y,x) < r\epsilon \ .$$

Now

$$|f(y) - f(x)| = r/r - \rho(x,z)/r < r\epsilon/r = \epsilon \ .$$