SOME THOUGHTS ON THE ABSOLUTE CONVERGENCE OF A TRIGONOMETRIC SERIES

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Although a major portion of this article is devoted to the development of a trigonometric series that fails to converge absolutely on $[0, 2\pi]$ but nevertheless is the Fourier Series of a continuous function, for the benefit of those readers who might have lost touch with Fourier Series, I will provide some definitions and various facts related to convergence of Fourier Series without going into proofs. The study of Fourier Series is an area that often remains completely unfamiliar to many of our undergraduates unless they are in a program that requires courses such as PDE or applied mathematical analysis/advanced engineering mathematics, in order to graduate. It is my understanding that a large number of this publication's readers are students and instructors of undergraduate mathematics. Even though I am focusing on one aspect of the convergence of Fourier Series in this article, this project gave me the opportunity to learn many things about Fourier Series that I had not known or had forgotten. I hope the readers will benefit from the small exposure to Fourier Series which this article provides and the subsequent interest in learning more about Fourier Series which this might possibly evoke. Some definitions and related results:

(1) A trigonometric series is of the form

$$\sum_{n=-\infty}^{\infty} C_n e^{inx}$$

where $C_n = a_n + ib_n$ $(n = 0, \pm 1, \ldots)$ are complex numbers. This also can be written in the form

$$A_0 + \sum_{n=1}^{\infty} (A_n \cos{(nx)} + B_n \sin{(nx)}) \ ,$$

where

$$A_0 = 2C_0, A_n = (a_n + a_{-n}) + i(b_n + b_{-n}),$$

$$B_n = (b_{-n} - b_n) + i(a_n - a_{-n}) \ .$$

If C_0 is real and $C_{-n} = \overline{C}_n$ for $n = 1, 2, \ldots,$ then A_0, A_n and B_n will all be real.

The series (1) is said to converge on $[0, 2\pi]$ if the sequence of partial sums $\{S_n(x)\}$ converges on $[0, 2\pi]$, where

$$S_n(x) = \sum_{m=-n}^{n} C_m e^{imx} \ , \ n = 1, 2, \ldots .$$

The trigonometric series (1) is said to be the Fourier Series of an integrable function f on $[0, 2\pi]$, with period 2π, if, for each n,

$$C_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt \ .$$

Many questions concerning a trigonometric series arise – questions about when a trigonometric series is the Fourier Series of a function, when does the Fourier Series of a function converge, when does it converge to the function itself, and the various types of convergence of a trigonometric series such as pointwise convergence, absolute convergence, and uniform convergence. A great number of these questions have been dealt with in research articles that have been published over the years.

As I had indicated earlier, I am focusing on the development, by Rudin and Shapiro (see [1], section 6), of an interesting trigonometric series that fails to converge absolutely on $[0, 2\pi]$ but nevertheless is the Fourier Series of a continuous function. Amazingly, the justification of many of the facts about this trigonometric series involves simple techniques such as 'Mathematical Induction' and well-known results about uniform convergence of series of functions, such as the 'Weierstrass M-test.' (see [3], Chapter 11, section 11-4)

The trigonometric series (1) is said to be absolutely convergent if the series

$$\sum_{n=-\infty}^{\infty} |C_n|$$

converges.

I will now state a result (see [2], Chapter I, Section 12) which will play an important role later in the discussion: "If a trigonometric series has a subsequence of partial sums that converges uniformly to function f on $[0, 2\pi]$, then the trigonometric series is the Fourier Series of this continuous function f."

The construction of a trigonometric series of the form

$$\sum_{n=1}^{\infty} C_n e^{int} \quad (C_{-n} = 0 \;\text{ for }\; n \geq 0)$$

that is the Fourier Series of a continuous function f on $[0, 2\pi]$ although

$$\sum_{n=1}^{\infty} |C_n|$$

fails to converge is done in several steps, as follows:

Define trigonometric polynomials $\{P_m(t)\}_{m=0}^{\infty}$ and $\{Q_m(t)\}_{m=0}^{\infty}$ on $[0, 2\pi]$, inductively by letting

$$P_0(t) = 1 = Q_0(t) \;\text{ and for } m \geq 0 \;,$$

$$P_{m+1}(t) = P_m(t) + e^{i2^m t} Q_m(t) \;\text{ and}$$

$$Q_{m+1}(t) = P_m(t) - e^{i2^m t} Q_m(t) \;.$$

We make the following observations (A) through (D) about the trigonometric polynomials $P_m(t)$ and $Q_m(t)$:

(A) Claim. $P_m(t)$ and $Q_m(t)$ are continuous on $[0, 2\pi]$ with

$$|P_m(t)| \leq 2^{(\frac{m+1}{2})} \;\text{ and }\; |Q_m(t)| \leq 2^{(\frac{m+1}{2})} \;.$$

$\underline{\text{Proof}}$. Clearly $P_m(t)$ and $Q_m(t)$ are continuous. Note

$$|P_m(t)|^2 + |Q_m(t)|^2 = P_m(t)\overline{P_m(t)} + Q_m(t)\overline{Q_m(t)}$$

(where $\overline{(\qquad)}$ is the complex conjugate of $(\qquad)$.)

$$= (P_{m-1} + e^{i2^{m-1}t}Q_{m-1})(\overline{P_{m-1}} + e^{-i2^{m-1}t}\overline{Q_{m-1}})$$

$$+ \quad (P_{m-1} - e^{i2^{m-1}t}Qm-1)(\overline{P_{m-1}} - e^{-i2^{m-1}t}\overline{Q_{m-1}})$$

$$= 2(|1P_{m-1}|^2 + |Q_{m-1}|^2)$$

$$= 2^2(|P_{m-2}|^2 + |Q_{m-2}|^2)$$

$$= \cdots = 2^m(|P_0|^2 + |Q_0|^2) = 2^{m+1} \ .$$

Therefore, $|P_m(t)|^2 \leq 2^{(\frac{m+1}{2})}$ and $|Q_m(t)|^2 \leq 2^{(\frac{m+1}{2})}$.

(B) For each $m \geq 0$,

$$\hat{P}_{m+1}(n) = \hat{P}_m(n) \ \text{ for } \ 0 \leq n < 2^m$$

where for any trigonometric polynomial

$$S_m(t) = \sum_{n=-m}^{m} C_n e^{int} \ ,$$

$$\hat{S}_m(n) = C_n = \frac{1}{2\pi}\int_0^{2\pi} S_m(t)e^{-int}dt \ , \quad -m \leq n \leq m \ .$$

To prove this, first note that by the way P_m and Q_m are defined $\hat{P}_m(n) = 0 = \hat{Q}_m(n)$ for $n < 0$. Now,

$$\hat{P}_{m+1}(n) = \frac{1}{2\pi}\int_0^{2\pi} P_m(t)e^{-int}dt + \frac{1}{2\pi}\int_0^{2\pi} Q_m(t)e^{-i(n-2^m)t}dt = \hat{P}_m(n) + 0$$

since $n - 2^m < 0 \Rightarrow Q_m(n - 2^m) = 0$ by the preceding remark.

(C) Defining the degree of the trigonometric polynomial

$$S_m(t) = \sum_{n=-m}^{m} C_n e^{int}$$

as the largest non-negative integer n for which $|C_n| + |C_{-n}| \neq 0$, we claim that the degree of this $P_m(t)$ and of $Q_m(t)$ is $2^m - 1$, for each $m \geq 0$. This can be proved using 'Mathematical Induction' as follows:

Clearly, when $m = 0$, $P_0(t) = Q_0(t) = 1$ so degree $P_0 = 0 =$ degree Q_0. Suppose, for some $m > 0$, degree $P_m = 2^m - 1 =$ degree Q_m, so that,

$$P_m(t) = \sum_{n=0}^{2^m-1} \alpha_n e^{int} \quad , \quad Q_m(t) = \sum_{n=0}^{2^m-1} \beta_n e^{int} \ .$$

Then, since,

$$P_{m+1}(t) = P_m(t) + e^{i2^m t} Q_m(t) = \left(\sum_{n=0}^{2^m-1} \alpha_n e^{int} \right) + e^{i2^m t} \left(\sum_{n=0}^{2^m-1} \beta_n e^{int} \right)$$

$$= \sum_{n=0}^{2^m-1} \alpha_n e^{int} + \sum_{n=0}^{2^m-1} \beta_n e^{i(n+2^m)t}$$

$$= \sum_{n=0}^{2^m-1} \alpha_n e^{int} + \sum_{k=2^m}^{2^{m+1}-1} (\beta_{k-2^m}) e^{ikt}$$

and similarly,

$$Q_{m+1}(t) = \sum_{n=0}^{2^m-1} \alpha_n e^{int} + \sum_{k=2^m}^{2^{m+1}-1} \big(\beta_{k-2^m}\big) e^{ikt} \ ,$$

it follows that degree $P_{m+1} = 2^{m+1} - 1 =$ degree Q_{m+1}.

(D) It now follows, from (B), (C) and the definition of P_m and Q_m, that there exists a sequence $\{a_n\}_{n=0}^{\infty}$ where $a_n = \pm 1$ for each $n \geq 0$, such that

$$P_m(t) = \sum_{n=0}^{2^m-1} a_n e^{int}$$

for all $m \geq 0$.

(E) We now define a sequence of trigonometric polynomials

$$\{T_m(t)\}_{m=1}^{\infty} \ \text{ as } \ T_m(t) = P_m(t) - P_{m-1}(t) \ .$$

Note that $T_m(t)$ can be written as $T_m(t) = e^{i2^{m-1}t} Q_{m-1}(t)$ and also as

$$T_m(t) = \sum_{n=2^{m-1}}^{2^m - 1} a_n e^{int} \ .$$

Clearly, the degree of $T_m(t)$ is $2^m - 1$.

(F) Claim. The sum of the absolute values of the terms of

$$T_m(t) \ \text{ is } \ 2^{m-1} \ \text{ and } \ |T_m(t)| = |Q_{m-1}(t)| \leq 2^{\frac{m}{2}} \ .$$

Proof. Since

$$T(t) = \sum_{n=2^{m-1}}^{2^m - 1} a_n e^{int}$$

where $a_n = \pm 1$, it is clear that

$$\begin{aligned} \sum_{n=2^{m-1}}^{2^m - 1} |a_n e^{int}| &= 2^m - 1 - 2^{m-1} + 1 \\ &= 2^{m-1} \ . \end{aligned}$$

Also

$$T_m(t) = e^{i2^{m-1}t} Q_{m-1}(t) \Rightarrow |T_m(t)| = |Q_{m-1}(t)| \leq 2^{\frac{m}{2}} \ ,$$

by A. We now define a trigonometric series as follows: Consider:

$$\sum_{n=1}^{\infty} 2^{-n} T_n(t) \ \underline{\qquad} (*) \ .$$

Substituting

$$\sum_{k=2^{n-1}}^{2^n-1} a_k e^{ikt}$$

in $(*)$, we obtain a trigonometric series

$$\sum_{n=1}^{\infty} C_n e^{int} \ , \quad \underline{\qquad}(**)$$

where for each $n \geq 1$, $C_n = 2^{-k} a_n$, k being the integer satisfying $2^{k-1} \leq n \leq 2^k - 1$.

(G) It can be easily verified that the sequence of partial sums of $(*)$ is simply the subsequence $\{S_{2^n-1}(t)\}_{n=1}^{\infty}$ of the sequence of partial sums of the trigonometric series $(**)$.

(H) <u>Claim</u>. $(*)$ converges uniformly to a function $f(t)$ on $[0, 2\pi]$.

<u>Proof</u>.

$$|2^{-n} T_n(t)| \leq 2^{-n} 2^{\frac{n}{2}} \qquad \text{(by F)}$$

$$= 2^{-\frac{n}{2}} \ .$$

Since the geometric series

$$\sum_{n=1}^{\infty} 2^{-\frac{n}{2}}$$

converges, Weierstrass's M-test now yields the conclusion. Clearly f is continuous on $[0, 2\pi]$.

(I) By (G) and a result stated earlier in this article, we now conclude that $(**)$ is the Fourier Series of this function $f(t)$ in (H).

(J) <u>Claim</u>. For the Fourier Series $(**)$,

$$\sum_{n=1}^{\infty} |C_n|$$

diverges.

Proof.

$$\sum_{n=1}^{\infty} |C_n| = \sum_{n=1}^{\infty} 2^{-k}(2^k - 2^{k-1}) \quad \text{(by the definition of } (**))$$

$$= \sum_{n=1}^{\infty} \frac{1}{2} .$$

In conclusion, I would like to say that the slick argument that is required in justifying the claims and conclusions in steps (G) through (J) is one reason why I feel this article is worth sharing with other mathematicians, although I am concentrating on one aspect of Fourier Series rather than a general discussion of Fourier Series.

References

1. Y. Katznelson, *An Introduction to Harmonic Analysis*, John Wiley & Sons, 1968.
2. Z. Zygmund, *Trigonometric Series*, Cambridge University Press, Volume I, 1968.
3. Saxena & Shah, *Introduction to Real Variable Theory*, Intext Educational Publishers, 1972.