SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the editor.

6. *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

Prove

$$\sum_{n\le x}\frac{1}{3n-2}=\frac{1}{3}\log(3x-2)+\frac{1}{6}\log 3+\frac{\pi}{6\sqrt{3}}+\frac{\gamma}{3}+O\Big(\frac{1}{x}\Big)\ ,$$

where log is the natural log and γ is Euler's constant.

Comment by Don Redmond, Southern Illinois University at Carbondale, Carbondale, Illinois.

The problem as given is to find an asymptotic expansion for

$$\sum_{n\le x}\frac{1}{3n-2}\ .$$

If we look in Ramanujan's notebooks (B.C. Berndt, Ramanujan's Notebooks, part I, p. 185) we find the following in Chapter 8, Entry 7. If x is a positive integer and a and b are arbitrary complex numbers, then

$$\Psi\left(\frac{a}{b}+x+1\right)-\Psi\left(\frac{a}{b}+1\right)=b\sum_{k=1}^{x}\frac{1}{a+bk}\ .$$

Here

$$\Psi(z)=\frac{\Gamma'}{\Gamma}(z)\ .$$

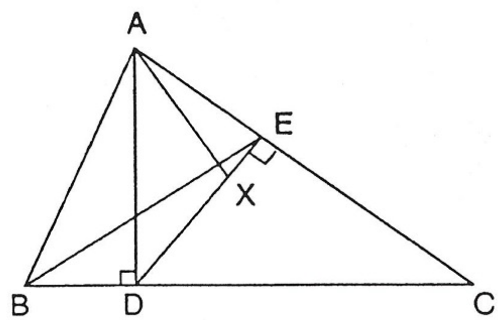
In Abramowitz and Stegun (Handbook of Mathematical Functions, p. 259) we find an asymptotic expansion for $\Psi(z)$ so that we may generate a general asymptotic expansion for

$$\sum_{n \le x} \frac{1}{a + bn}$$

with a and b complex numbers.

14. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

In triangle ABC, AD is an altitude (with D lying on segment BC). $DE \perp AC$ with E lying on AC. X is a point on segment DE such that $\frac{EX}{XD} = \frac{BD}{DC}$. Prove that $AX \perp BE$.



Solution by Tran van Thuong, Missouri Southern State College, Joplin, Missouri.

From D, we draw $DM \parallel BE$, then we have

(1) $$\frac{BD}{DC} = \frac{EM}{MC} \ .$$

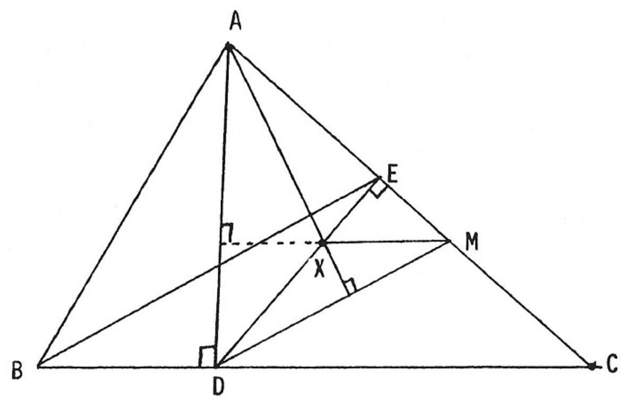
By assumption, we have

(2) $$\frac{BD}{DC} = \frac{XE}{XD} \ .$$

Compare (1) and (2), we see that

$$\frac{EM}{MC} = \frac{XE}{XD} \ .$$

This shows that $XM \parallel DC$. Since $AD \perp BC$, we have $XM \perp AD$. Now looking at $\triangle AMD$, we see that $DE \perp AM$ and $XM \perp AD$; it follows that $AX \perp DM$. Therefore $AX \perp BE$ (since $BE \parallel DM$). This completes the proof.



Also solved by Russell Euler, Northwest Missouri State University, Maryville, Missouri and the proposer.

15. *Proposed by Leonard L. Palmer, Southeast Missouri State University, Cape Girardeau, Missouri.*

Let F_n denote the nth Fibonacci number ($F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-2} + F_{n-1}$ for $n > 2$) and let L_n denote the nth Lucas number ($L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-2} + L_{n-1}$ for $n > 2$). Find all x such that $F_x + L_x \equiv 0 \pmod 4$ and verify.

Composite solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin, Russell Euler, Northwest Missouri State University, Maryville, Missouri, and Alex Necochea, University of Texas-Pan American, Edinburg, Texas.

We shall show that $F_x + L_x \equiv 0 \pmod 4$ if and only if $x = 3n - 1$ for some positive integer n. Our solution will use the following known result:

(∗) $F_m | F_n$ if and only if $m|n$ for $m \geq 2$.

(For a proof of this theorem, see pp. 334–336 of Burton; Elementary Number Theory (Second Edition); Wm. C. Brown Publishers; Dubuque, Iowa; 1989.) We will also employ the following lemma which is easily proved by using the Binet forms for L_n and F_n.

Lemma. $L_k = F_{k-1} + F_{k+1}$ for each positive integer k.

Now let x be a positive integer. Then from the Lemma,

$$\begin{aligned} F_x + L_x &= F_x + (F_{x-1} + F_{x+1}) \\ &= (F_x + F_{x-1}) + F_{x+1} \\ &= F_{x+1} + F_{x+1} \\ &= 2F_{x+1} \ . \end{aligned}$$

Hence by (∗),

$$\begin{aligned} F_x + L_x \equiv 0 \pmod 4 \text{ if and only if } & 2|F_{x+1} \\ \text{iff } & F_3 | F_{x+1} \\ \text{iff } & 3|(x+1) \\ \text{iff } & x = 3n - 1 \text{ for some positive integer } n \ . \end{aligned}$$

Also solved by J.E. Chance, University of Texas-Pan American, Edinburg, Texas, Bill Wynns and Dale Woods (jointly) Central State University, Edmond, Oklahoma, Donald Skow, University of Texas-Pan American, Edinburg, Texas, and the proposer.

Alex Necochea remarked that an analogous calculation shows that $F_x - L_x = -2F_{x-1}$ for all integers $x \geq 1$, where $F_0 = 0$.

Again, $2 = F_3 | F_{x-1}$ iff $3 | x-1$. Therefore, one has that $F_x - L_x \equiv 0 \pmod 4$ if and only if $x \equiv 1 \pmod 3$. Putting these two results together, one has that

$$F_x \pm L_x \equiv 0 \pmod 4 \text{ iff } x \equiv \mp 1 \pmod 3 .$$

16. *Proposed by Mark Ashbaugh, University of Missouri, Columbia, Missouri.*

Consider the $2n \times 2n$ matrix T_n defined as the skew-symmetric matrix for which each entry in the first n subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is -1. For example,

$$T_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$T_2 = \begin{pmatrix} 0 & -1 & -1 & 1 \\ 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{pmatrix} .$$

Find $\det T_n$ for all positive integers n.

Solution by the proposer.

Let S be the $n \times n$ skew-symmetric matrix with all entries below the main diagonal equal to 1. Then

$$T_n = \begin{pmatrix} S & -I - S \\ I - S & S \end{pmatrix}$$

and thus (adding row 2 to row 1 and then adding column 1 to column 2), we have

$$\begin{aligned} \det T_n &= \det \begin{pmatrix} S & -I - S \\ I - S & S \end{pmatrix} \\ &= \det \begin{pmatrix} I & -I \\ I - S & S \end{pmatrix} \\ &= \det \begin{pmatrix} I & 0 \\ I - S & I \end{pmatrix} \\ &= 1 . \end{aligned}$$

The transformations that we have performed on the partitioned determinants above are easily justified in terms of elementary operations on determinants.

Also solved by Alex Necochea, University of Texas-Pan American, Edinburg, Texas and Sam Cazares and Dale Woods (jointly), Central State University, Edmond, Oklahoma.

Comment by the proposer.

This matrix figured in Problem B-5 in the 1988 Putnam exam. There it sufficed to know that $\det T_n \neq 0$ and this is relatively easy to see by working modulo 2. Here we ask for the explicit evaluation of $\det T_n$.