INNER PLETHYSM IN THE REPRESENTATION RING OF THE GENERAL LINEAR GROUP*

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The investigation of plethysms (inner and outer) in the representation theory of finite classical groups has been one of the important outstanding problems in the representation theory of the symmetric group [5], [7] and [8]. The fundamental theorem of the representation theory of the symmetric group has been more or less known since the origins of the subject with Frobenius at the turn of the century. This theorem states there is an isomorphism between the representation ring of the symmetric groups S_n and the ring of symmetric polynomials in an infinite number of variables. But for various reasons this isomorphism in its pure form seems not to have appeared until Atiyah [2] introduced the Steenrod power operations in K-Theory around 1966. In [2] Atiyah described how to use the complex representations of the symmetric group S_k, to

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define and investigate operations in K-Theory. Atiyah's main technical tool is the notation of λ-ring introduced by Grothendieck [3] in 1956 in an algebraic-geometric context. The notation of λ-ring has been used by Knutson [6], to study the fundamental theorem of the representation theory of the symmetric group which has been translated into that of plethysms by Hoffman [4], Uehara and myself [1] and [9]. The main purpose of this paper is to define and investigate the inner plethysm in the representation ring $R(A_n)$, where $A_n = GL(n, K)$ in the nth general linear group over a finite field K. In the case $A_n = S_n$, the symmetric group, Hoffman [4] investigated the inner and outer plethysms in the frame work of τ-rings. The authors of [1] and [9] studied the outer plethysms for $A_n = S_n$, $A_n = GL(n, K)$ and $A_n = [G]S_n$, the wreath product of a finite group G by the symmetric group S_n.

Let $A_n = GL(n, K)$ be the nth general linear group over a finite field K. The wreath product $[A_n]S_n$ of A_n by the symmetric group S_k (the usual notation for $[A_n]S_k$ is $A_n \wr S_k$) is the set $A_n^k \times S_k$ with a multiplication defined by $(a_1, \ldots, a_k; \sigma)(a'_1, \ldots, a'_k; \sigma') = (a_1 a'_{\sigma^{-1}(1)}, \ldots, a_k a'_{\sigma^{-1}(k)}; \sigma\sigma')$ where $a_i, a'_i \in A_n$ for $k \geq i \geq 1$

and $\sigma, \sigma' \in S_k$. For a representation M of A_n and for $k \geq 1$, the kth tensor product of M, $M^{\otimes k} = M \otimes M \otimes \cdots \otimes M$ (k factors) is a representation of $[A_n]S_k$ with a group action given by $(a_1, \ldots, a_k; \sigma)(m_1 \otimes m_2 \otimes \cdots \otimes m_k) = a_1 m_{\sigma^{-1}(1)} \otimes \cdots \otimes a_k m_{\sigma^{-1}(k)}$ for any $(a_1, \ldots, a_k; \sigma) \in [A_n]S_k$ and $m_i \in M$. In what follows $\otimes$ means $\otimes_c$ and we interpret $M^{\otimes 0}$ as the A_n-representation C, on which A_n acts trivially.

For any finite group G let $R(G)$ be the Grothendieck representation group of G, $R(G)$ is the free abelian group generated by the isomorphism classes of irreducible complex representations of G. It is a ring with respect to the tensor product. For every integer $k \geq 1$ we have a map

$$\otimes k : R(A_n) \to R([A_n]S_k)$$

defined by $\otimes k([M]) = [M^{\otimes k}]$, where $[M] \in R(A_n)$ is the class of M.

First we are going to show $\otimes k$ is well defined (compare Atiyah [2], proposition 2.2). Let G be any finite group and consider the semiring $M(G) = \{(M, N) | M, N \text{ are } G\text{−modules } \}$ with addition and multiplication defined by

$$(M,N)+(M',N')=(M\oplus M',N\oplus N')$$

and

$$(M,N).(M',N')=(M\otimes M'\oplus N\otimes N',M\otimes N'\oplus M'\otimes N)\ .$$

We define an equivalence relation $\sim$ on $M(G)$ by $(M,N)\sim(M',N')$ if and only if $M\oplus N'\simeq M'\oplus N$. We denote by $[(M,N)]$ the equivalence class of $(M,N)\in M(G)$. Then $\overline{R}(G)=M(G)/\sim$ is a ring with $0=[(D,D)]$ and $-[(M,N)]=[(N,M)]$. It is clear from the construction that the map $h:\overline{R}(G)\to R(G)$ defined by $h([(M,n)])=[M]-[N]$ is a ring isomorphism. For each integer $k\geq 1$, we define a map $\triangle_k:M(A_n)\to M([A_n]S_k)$ by $\triangle_k(M,N)=(M,N)^k$.

<u>Lemma 1</u>. The map $\triangle_k$ is compatible with the equivalence relation $\sim$ on $M(A_n)$.

<u>Proof</u>. We have to show that if D is any A_n-module and $(M,N)\in M(A_n)$ then $\triangle_k(M,N)\sim\triangle_k(M\oplus D,N\oplus D)$. This

can be proved by induction on k. If $k = 1$,

$$\triangle_1(M \oplus D, N \oplus D) = (M \oplus D, N \oplus D) = (M, N) + (D, D) = (M, N) \; .$$

Assume the hypothesis is true for $k - 1$, then

$$\begin{aligned}
\triangle_k(M \oplus D, N \oplus D) &= (M \oplus D, N \oplus D)^{k-1}.(M \oplus D, N \oplus D) \\
&= (M, N)^{k-1}((M, N) + (D, D)) \\
&= (M, N)^{k-1}(M, N) = \triangle_k(M, N) \; .
\end{aligned}$$

Now consider the diagram

$$\begin{array}{ccc}
M(A_n) & \xrightarrow{\triangle_k} & M([A_n]S_k) \\
\Big\downarrow P & & \Big\downarrow P \\
\overline{R}(A_n) & \xrightarrow{\overline{\otimes} k} & \overline{R}([A_n]S_k) \\
\Big\downarrow h & & \Big\downarrow h \\
R(A_n) & \xrightarrow{\otimes k} & R([A_n]S_k)
\end{array}$$

where $\overline{\otimes} k$ is the map induced by $\triangle_k$ and P is the projection map.

$$\begin{aligned}
h \circ p \circ \triangle_k(M, 0) &= h \circ p(M, 0)^k = h \circ p(M^{\otimes k}, 0) \\
&= h([(M^{\otimes k}, 0)] = [M^{\otimes k}] = \otimes k[M] \; .
\end{aligned}$$

Similarly $\otimes k \circ h \circ p(M,0) = \otimes k[M]$. Thus it follows that the diagram commutes and $\otimes k$ is also induced by $\triangle_k$, and hence the map $\otimes k$ is well defined.

Before we state the next result we recall the following.

Definition. Let H be a subgroup of a finite group G and M is a complex representation of H the representation of G induced by M is given by $\mathrm{Ind}_H^G M = CG \otimes_{CH} M$.

Lemma 2. If $(M,N) \in M(A_n)$, then for any $k \geq 1$,

$$\triangle_k(M,N) = \Big(\sum_{\substack{i=0 \\ i\ even}}^{k} \mathrm{Ind}_{[A_n]S_{k-i}\times[A_n]S_i}^{[A_n]S_k}(M^{\otimes(k-i)} \otimes N^{\otimes i}),$$

$$\sum_{\substack{j=1 \\ j\ odd}}^{k} \mathrm{Ind}_{[A_n]S_{k-j}\times[A_n]S_j}^{[A_n]S_k}(M^{\otimes(k-j)} \otimes N^{\otimes j})\Big) .$$

Proof. The proof is by induction on k. If $k=1$, this is evident. Assume that the hypothesis is true for all integers $m \leq k$. Then we have

$$\triangle_{k+1}(M,N) = (M,N)^k(M,N)$$

$$= \Big(\sum_{\substack{i=0 \\ i \ even}}^{k} \mathrm{Ind}_{[A_n]S_{k-i}\times[A_n]S_i}^{[A_n]S_k}(M^{\otimes(k-i)} \otimes N^{\otimes i}),$$

$$\sum_{\substack{j=1 \\ j \ odd}}^{k} \mathrm{Ind}_{[A_n]S_{k-j}\times[A_n]S_j}^{[A_n]S_k}(M^{\otimes(k-j)} \otimes N^{\otimes j})\Big)(M,N)\Big)$$

$$= (\sum_{\substack{i=0 \\ i \ even}}^{k} \mathrm{Ind}_{[A_n]S_{k-i}\times[A_n]S_i}^{[A_n]S_k}(M^{\otimes(k-i)} \otimes N^{\otimes i}) \otimes M \oplus$$

$$\sum_{\substack{j=1 \\ j \ odd}}^{k} \mathrm{Ind}_{[A_n]S_{k-j}\times[A_n]S_j}^{[A_n]S_k}(M^{\otimes(k-j)} \otimes N^{\otimes j}) \otimes N,$$

$$\sum_{\substack{i=0 \\ i \ even}}^{k} \mathrm{Ind}_{[A_n]S_{k-i}\times[A_n]S_i}^{[A_n]S_k}(M^{\otimes(k-i)} \otimes N^{\otimes i}) \otimes N \oplus M \otimes$$

$$\sum_{\substack{j=1 \\ j \ odd}}^{k} \mathrm{Ind}_{[A_n]S_{k-j}\times[A_n]S_j}^{[A_n]S_k}(M^{\otimes(k-j)} \otimes N^{\otimes j}))$$

$$= (\sum_{\substack{i=0 \\ i \ even}}^{k+1} \mathrm{Ind}_{[A_n]S_{k+1-i}\times[A_n]S_i}^{[A_n]S_{k+1}}(M^{\otimes(k+1-i)} \otimes N^{\otimes i}),$$

$$\sum_{\substack{j=1 \\ j \ odd}}^{k+1} \mathrm{Ind}_{[A_n]S_{k+1-j}\times[A_n]S_j}^{[A_n]S_{k+1}}(M^{\otimes(k+1-j)} \otimes N^{\otimes j})\big) \ .$$

Corollary 3. For any $[M]-[N] \in R(A_n)$ and $k \geq 1$,

$$\otimes k([M]-[N]) = \sum_{i=0}^{k}(-1)^i[\mathrm{Ind}_{[A_n]S_{k-i}\times[A_n]S_i}^{[A_n]S_k}(M^{\otimes(k-i)} \otimes N^{\otimes i})] \ .$$

Proof. Since $\triangle_k$ induces $\otimes k$, apply $h \circ p$ to $\triangle_k(M,N)$ as given in the Lemma and we are done.

By construction $[A_n]S_k$ is the semi-direct product $A_n^k \times_\theta S_k$, where $\theta : S_k \to \text{ Aut } (A_n^k)$ is a group homomorphism given by

$$\theta(\sigma)((a_1,\ldots,a_k)) = (a_{\sigma^{-1}(1)},\ldots,a_{\sigma^{-1}(k)})$$

for $\sigma \in S_k$, $a_i \in A_n$. In other words, the short exact sequence

$$1 \to A_n^k \to [A_n]S_k \to S_k \to 1$$

is split by the obvious maps $\alpha : S_k \to [A_n]S_k$ where $\alpha(\sigma) = (1,1,\ldots,1;\sigma)$. Also the image of A_n in $[A_n]S_k$ as the diagonal subgroup

$$\{(a,a,\ldots,a)|a \in A_n\}$$

of A_n^k, commutes elementwise with the image of S_k thus we get an embedding $\psi : A_n \times S_k \to [A_n]S_k$. Hereafter, $A_n \times S_k$ is considered as a subgroup of $[A_n]S_k$.

Definition. By the inner plethysm T_ϕ associated with an element $\phi \in R_c^*(S_k) = \text{ Hom }_c(R(S_k), C)$, we mean the map

$$\begin{array}{ccc} T_\phi : R(A_n) & \xrightarrow{R} & (A_n) \otimes C \simeq R(A_n) \\ \big\downarrow{\scriptstyle \otimes k} & & \\ R(A_n) \otimes R(S_k) & & \end{array}$$

defined by $T_\phi = (1 \otimes \phi) \circ (\otimes k)$. In the sequel for any $[M] \in R(A_n)$ we denote $T_\phi([M])$ by $\phi([M])$, if no confusion arises. For a partition $\pi = \{1^{\pi 1}, 2^{\pi 2}, \ldots, k^{\pi k}\}$ of k (in notation $\pi \vdash k$), let $S_\pi = S_1^{\pi 1} \times S_2^{\pi 2} \times \cdots \times S_k^{\pi k}$. Then a trivial representation and a sign representation of S_π are denoted by 1_{S_π} and Alt S_π respectively. Let $\rho_\pi = [\text{Ind}_{S_\pi}^{S_k} 1_{S_\pi}]$ and $\eta_\pi = [\text{Ind}_{S_\pi}^{S_k} \text{ Alt } S_\pi]$.

Lemma 3.

$$\{\rho_\pi | \pi \vdash k\} \text{ and } \{\eta_\pi | \pi \vdash k\} \text{ are bases for } R(S_k)\ .$$

This fact is known, for example, see Knutson [6]. Let us consider the elements λ_π and σ_π in $R^*(S_k)$ defined by

$$\lambda_\pi([M]) = \begin{cases} 1, & \text{if } M = \text{ Alt } S_\pi; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\sigma_\pi([M]) = \begin{cases} 1, & \text{if } M = 1_{S_\pi}; \\ 0, & \text{otherwise.} \end{cases}$$

Define a map $\mu : R(S_k) \to R^*(S_k)$ by $\mu([M])([N]) =< M, N >$ for $[M], [N] \in R(S_k)$ where $<, >$ is the Schur inner product in $R(S_k)$. Then it is known that μ is a ring isomorphism and $\mu(\rho_\pi) = \sigma_\pi$ and $\mu(\eta_\pi) = \lambda_\pi$, where $\pi \vdash k$. If E is an A_n-representation and V is an S_k-representation, then $\mathrm{Hom}_{S_k}(V, E^{\otimes k})$ can be considered as an A_n-representation when a group action is defined by $a \bullet f = a^{\otimes k} \circ f$ for all $f \in \mathrm{Hom}_{S_k}(V, E^{\otimes k})$ and $a \in A_n$. It is well known that if $\{V_\pi | \pi \vdash k\}$ is a complete set of irreducible S_k-representations then there exists an $A_n \times S_k$-isomorphism.

$$\theta : \sum_{\pi \vdash k} \mathrm{Hom}_{S_k}(V_\pi, E \otimes k) \otimes V_\pi \to E^{\otimes k}$$

defined by $\theta(f \otimes x) = f(x)$ for $f \in \mathrm{Hom}_{S_k}(V_\pi, E^{\otimes k})$ and $x \in V_\pi$.

Theorem 5. For any $\lambda_\tau \in R^*(S_k)$ with $\tau \vdash k$ and for any A_n-module M, we have

$$\lambda_\tau([M])[\mathrm{Hom}_{S_k}(\mathrm{Ind}_{S_\tau}^{S_k} \mathrm{Alt}\ S_\tau, M^{\otimes k})]$$

Proof. First consider the $A_n \times S_k$ decomposition

$$M^{\otimes k} \simeq \sum_{\pi \vdash k} \mathrm{Hom}_{S_k}(V_\pi, M^{\otimes k}) \otimes V_\pi \ .$$

Then by definition

$$\lambda_\tau([M]) = (1 \otimes \lambda_\tau).(\otimes k)([M])$$

$$= (1 \otimes \lambda_\tau)([M^{\otimes k}])$$

$$= \sum_{\pi \vdash k} \mathrm{Hom}_{S_k}(V_\pi, M^{\otimes k}) \lambda_\tau([V_\pi])$$

$$= [\mathrm{Hom}_{S_k}(\sum \lambda_\tau([V_\pi])V_\pi, M^{\otimes k})]$$

However,

$$\sum_{\pi \vdash k} \lambda_\tau([V_\pi])V_\pi = \sum_{\pi \vdash k} \mu_\tau(\eta_\tau)([V_\pi])V_\pi$$

$$= \sum_{\pi \vdash k} < \mathrm{Ind}_{S_\tau}^{S_k} \mathrm{Alt}\ S_\tau, V_\pi > V_\pi$$

$$= \mathrm{Ind}_{S_\tau}^{S_k} \mathrm{Alt}\ S_\tau\ .$$

Hence we obtain $\lambda_\tau([M]) = [\mathrm{Hom}_{S_k}(\mathrm{Ind}_{S_\tau}^{S_k} \mathrm{Alt}\ S_\tau, M^{\otimes k})]$.

Theorem 6. For any partition $\tau = \{1^{\tau 1}, 2^{\tau 2}, \ldots, k^{\tau k}\} \vdash k$ and for any A_n-representation M, we have

$$\lambda_\tau([M]) = \lambda_1([M])^{\tau 1} \lambda_2([M])^{\tau 2} \cdots \lambda_k([M])^{\tau k}\ .$$

Proof. By the Frobenius reciprocity law we have

$$\mathrm{Hom}_{S_k}(\mathrm{Ind\ Alt}\ S_\tau, M^{\otimes k}) \simeq \mathrm{Hom}(\mathrm{Alt}\ S_\tau, \mathrm{Res}_{S_\tau}^{S_k} M^{\otimes k})$$

since

$$\mathrm{Alt}\ S_\tau \simeq (\mathrm{Alt}\ S_1)^{\otimes \tau 1} \otimes \cdots \otimes (\mathrm{Alt}\ S_k)^{\otimes \tau k}$$

and

$$\mathrm{Res}_{S_\pi}^{S^k}(M^{\otimes k})M^{\otimes\tau 1}\otimes(M^{\otimes 2})^{\otimes\tau 2}\otimes\cdots\otimes(M^{\otimes k})\otimes\tau k\ ,$$

we obtain

$$\mathrm{Hom}_{s_\tau}(\mathrm{Alt}\ _{S_\tau},\mathrm{Res}_{s_\tau}^{S_k}M^{\otimes k})\simeq\otimes_{i=1}^{k}(\mathrm{Hom}_{S_i}(\mathrm{Alt}\ S_i,M^{\otimes i}))^{\otimes\tau i}$$

By Theorem 5 we have

$$\begin{aligned}\lambda_\tau([M]) &= \mathrm{Hom}_{S_k}(\mathrm{Ind}_{S_\tau}^{S_k}\mathrm{Alt}\ S_\tau, M^{\otimes k}))]\\ &= \prod_{i=1}^{k}[\mathrm{Hom}_{S_i}((\mathrm{Alt}\ S_i,M^{\otimes i}))]^{\otimes\tau i}\\ &= \lambda_1([M])^{\tau 1}\lambda_2([M])^{\tau 2}\cdots\lambda_k([M])^{\tau k}\ .\end{aligned}$$

This completes the proof.

Theorem 7. For any $\sigma_\tau\in R^*(S_k)$ where $\tau=\{1^{\tau 1},\ldots,k^{\tau k}\}$ and any A_n-representation M, we have

$$\begin{aligned}\sigma_\tau([M]) &= [\mathrm{Hom}_{S_k}(\rho_\tau,M^{\otimes k})]\\ &= \sigma_1([M])^{\tau 1}\sigma_2([M])^{\tau 2}\cdots\sigma_k([M])^{\tau k}\end{aligned}$$

The proof is similar to that of Theorem 5 and Theorem 6.

To compute the inner plethysm associated with λ_k and σ_k for a general element $[M]-[N]\in R(A_n)$, we need the following lemmas:

Lemma 8. Let $H \subseteq G \subseteq S_n$ be groups and let N be a representation of H. Then $\mathrm{Hom}_G(\mathrm{Alt}\ G, \mathrm{Ind}_H^G N)$ and $\mathrm{Hom}_H(\mathrm{Alt}\ H, N)$ are isomorphic.

Proof. We construct a linear map

$$p : \mathrm{Hom}_G(\mathrm{Alt}\ G, \mathrm{Ind}_H^G N) \to \mathrm{Hom}_H(\mathrm{Alt}\ H, N)$$

and its inverse σ. Let $\{e = r_0, r_1, \ldots r_t\}$ be a complete set of coset representatives for G/H. Then

$$\mathrm{Ind}_H^G N \simeq N \oplus r_1 N \oplus \cdots \oplus r_t N \ .$$

If $U \in \mathrm{Hom}_G(\mathrm{Alt}\ G, \mathrm{Ind}_H^G N)$ then there are $n_i \in N_i$ such that

$$U(1) = n_0 + r_1 n_1 + \cdots + r_t n_t \ .$$

We let p be the linear map from C to N defined by $p(U)(1) = n_0$. p is an H-homomorphism because if $h \in H$, then

$$hp(U)(1) = hn_0 = \mathrm{sgn}(h) n_0 = p(u)(\mathrm{sgn}(h)) = p(f)(h1) \ .$$

We now construct σ. If $\omega \in \mathrm{Hom}_H(\mathrm{Alt}\ H, N)$ and $\omega(1) = n_0$, let σ be the linear map from C to $N \oplus r_1 N \oplus \cdots \oplus r_t N$ defined by

$$\sigma(\omega)(1) = \sum_{i=0}^{t} \mathrm{sgn}(r_i) r_i n_0 \ .$$

σ is a G-homomorphism because if $g \in G$, then

$$g\sigma(\omega)(1) = \sum_{i=0}^{t} \mathrm{sgn}(r_i) r_i n_0 \ .$$

Furthermore, since $\{gr_0, gr_1, \ldots gr_t\}$ is a set of coset representatives for G/H, there exist elements $h_0, \ldots, h_t \in H$ and there is a permutation τ of $\{0, \ldots, t\}$ such that $gr_i = r_\tau(i)h_i$. Hence,

$$\begin{aligned}
\sum_{i=0}^{t} \mathrm{sgn}(r_i) g r_i n_0 &= \sum_{i=0}^{t} \mathrm{sgn}(r_i) r_\tau(i) h_i n_0 \\
&= \sum_{i=0}^{t} \mathrm{sgn}(r_i)\mathrm{sgn}(h_i) r_\tau(i) n_0 \\
&= \sum_{i=0}^{t} \mathrm{sgn}(g)\mathrm{sgn}(r_\tau(i)) r_\tau(i) n_o \\
&= \sum_{i=0}^{t} \mathrm{sgn}(g)\mathrm{sgn}(r_i) r_i n_o \\
&= \mathrm{sgn}(g)\sigma(\omega)(1) = \sigma(\omega)(\mathrm{sgn}(g)) = \sigma(\omega)(g \bullet 1) \ .
\end{aligned}$$

We now show that $\sigma \circ p$ is the identity. Consider

$$U(1) = \sum_{i=1}^{t} r_i n_i$$

and

$$\sigma \circ p(U)(1) = \sum_{i=0}^{t} \mathrm{sgn}(r_i) r_i n_0 \ .$$

It suffices to show that $\mathrm{sgn}(r_k)n_0 = n_k$ for all k. Since U is a G-homomorphism,

$$r_k U(1) = U(r_k 1) = U(\mathrm{sgn}(r_k)) = \mathrm{sgn}(r_k)\sum_{i=0}^{t} r_i n_i \ .$$

On the other hand,

$$r_k U(1) = \sum_{i=0}^{t} r_k r_i n_i \ .$$

Hence $\mathrm{sgn}(r_k)r_k n_k = r_k n_0$ and $\mathrm{sgn}(r_k)n_k = 0$. The proof is complete, since it is obvious that $P \circ \sigma$ is the identity.

<u>Lemma 9</u>. Let $H \subseteq G \subseteq S_n$ be groups and let N be a representation of H. Then $\mathrm{Hom}_G(1_G, \mathrm{Ind}_H^G N)$ and $\mathrm{Hom}_H(1_H, N)$ are isomorphic.

<u>Proof</u>. Similar to the proof of Lemma 8.

<u>Theorem 10</u>. For any $[M] - [N] \in R(A_n)$, where M is assumed to have even grading and N to have odd grading, we have:

$$(i)\ \lambda_k([M] - [N]) = \sum_{i=0}^{k} (-1)^i \lambda_{k-i}([M])\sigma_i([N])$$

$$(ii)\ \sigma_k([M]) - [N]) = \sum_{i=0}^{k} (-1)^i \sigma_{k-i}([M])\lambda_i([N]) \ .$$

<u>Proof</u>. (i) By definition and since N have odd grading,

$$\lambda_k([M]-[N]) = (1\otimes\lambda_k)\circ(\otimes k)([M]-[N])$$

$$= (1\otimes\lambda_k)(\sum_{i=0}^{k}(-1)^i[\mathrm{Ind}_{[A_n]S_{k-i}\times[A_n]S_i}^{[A_n]S_k}(M^{\otimes(k-i)}\otimes N^{\otimes i})])$$

$$= \sum_{i=0}^{k}(-1)^i(1\otimes\lambda_k)[\mathrm{Ind}_{[A_n]S_{k-i}\times[A_n]S_i}^{[A_n]S_k}(M^{\otimes(k-i)}\otimes N^{\otimes i})]$$

$$= \sum_{i=0}^{k}(-1)^i[\mathrm{Hom}_{S_k}(\mathrm{Alt}\ S_k, \mathrm{Ind}_{[A_n]S_{k-i}\times[A_n]S_i}^{[A_n]S_k}(M^{\otimes(k-i)}\otimes N^{\otimes i}))]$$

$$= \sum_{i=0}^{k}(-1)^i[\mathrm{Hom}_{S_{k-i}\times S_i}(\mathrm{Alt}\ (S_{k-i}\times S_i), M^{\otimes(k-i)}\otimes N^{\otimes i})]$$

$$= \sum_{i=0}^{k}(-1)^i[\mathrm{Hom}_{S_{k-i}\times S_i}(\mathrm{Alt}\ S_{k-i}\otimes S_i, M^{\otimes(k-i)}\otimes N^{\otimes i})]$$

$$= \sum_{i=0}^{k}(-1)^i[\mathrm{Hom}_{S_{k-i}}(\mathrm{Alt}\ S_{k-i}M^{\otimes(k-i)})\otimes\mathrm{Hom}_{S_i}(\mathrm{Alt}\ S_i, N^{\otimes i})]$$

$$= \sum_{i=0}^{k}(-1)^i[\mathrm{Hom}_{S_{k-i}}(\mathrm{Alt}\ S_{k-i}M^{\otimes(k-i)})][\mathrm{Hom}_{s_i}(\mathrm{Alt}\ S_i, N^{\otimes i})]$$

$$= \sum_{i=0}^{k}(-1)^i[\mathrm{Hom}_{S_{k-i}}(\mathrm{Alt}\ S_{k-i}M^{\otimes(k-i)})][\mathrm{Hom}_{S_i}(1_{S_i}, N^{\otimes i})]$$

$$= \sum_{i=0}^{k}(-1)^i\lambda_{k-i}(([M]\sigma_i([N])\ .$$

The proof of (ii) is similar to (i). Hence, the proof is complete.

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