ON THE LOWER NEAR FRATTINI SUBGROUPS OF AMALGAMATED FREE PRODUCTS OF GROUPS I

Mohammad K. Azarian

University of Evansville

1. **Introduction.** In this paper, we briefly reintroduce the Frattini subgroup $\Phi(G)$, the lower near Frattini subgroup $\lambda(G)$, the upper near Frattini subgroup $\mu(G)$, the near Frattini subgroup $\psi(G)$, of a group G, as well as the amalgamated free products of groups. Also, we prove a generalization of a lemma by C. Y. Tang, and its exact analog for the lower near Frattini subgroups. Finally, we propose two questions for readers.

2. **Notation and Definitions.** Our notation will be standard.

Definition 1. An element g of a group G is a *nongenerator* of G if for every subset S of G such that $< S, g > = G$, then $< S > = G$.

Definition 2. The set of all nongenerators of a group G forms a characteristic subgroup called the *Frattini subgroup* of G, denoted by $\Phi(G)$. The intersection of all maximal proper subgroups of G coincides with $\Phi(G)$. If there are no maximal proper subgroups of G, then $\Phi(G) = G$.

For more results concerning $\Phi(G)$ see [9].

Definitions involving near Frattini subgroups are due to J. B. Riles [7].

Definition 3. An element g of a group G is a *near generator* of G if there is a subset S of G such that $|G :< S > |$ is infinite, but $|G :< g, S > |$ is finite.

Definition 4. An element g of a group G is a *non-near generator* of G if $S \subseteq G$ and $|G :< g, S > |$ is finite imply that $|G :< S > |$ is finite.

Definition 5. By [1, Proposition 1] the set of all non-near generators of a group G is a subgroup of G. This subgroup is called the *lower near Frattini* subgroup of G, denoted by $\lambda(G)$.

Note 1. $\lambda(G)$ is a characteristic subgroup of G, since every automorphism of G maps a non-near generator of G to a non-near generator of G.

Definition 6. A subgroup M of a group G is *nearly maximal* in

G if $|G:M|$ is infinite, but $|G:N|$ is finite, for every subgroup N of G properly containing M. That is, M is maximal with respect to being of infinite index in G.

Definition 7. The intersection of all nearly maximal subgroups of a group G is called the *upper near Frattini subgroup* of G, denoted by $\mu(G)$.

Note 2. $\mu(G)$ is a characteristic subgroup of G, since every automorphism of G permutes the nearly maximal subgroups of G among themselves.

Definition 8. Let G be any group. If $\lambda(G) = \mu(G)$, then their common value is called the *near Frattini* subgroup of G, denoted by $\psi(G)$.

Definition 9. If H is a subgroup of a group G, then

$$K(G,H) = \bigcap_{g\in G} g^{-1}Hg$$

is called the *core* of H in G. $K(G,H)$ is the unique largest normal subgroup of G contained in H.

3. **Amalgamated Free Products of Groups.** Free products of groups with amalgamated subgroups was published first

in 1927 by the German mathematician O. Schreier [8]. Hanna Neumann generalized Schreier's original results, and her work was published in two separate papers in 1948 [4] and 1949 [5]. B. H. Neumann in his 1954 paper [3], redefined, studied, and applied Schreier's and H. Neumann's results to a number of problems in group theory. In this paper we adopt the viewpoint of B. H. Neumann's paper and the following usages for free products of groups with amalgamations.

Let Γ be an indexing set of cardinality greater than one, and let G be a group with a set S of generators. Suppose that

$$S = \bigcup_{\gamma \in \Gamma} S_\gamma$$

is a union of subsets S_γ. Set $G_\gamma = < S_\gamma >$ for each γ, and let R_γ be a set of defining relations for G_γ. If

$$R = \bigcup_{\gamma \in \Gamma} R_\gamma$$

is a set of defining relations for G, then G is the *generalized free product* of the subgroups G_γ. Since S_α and S_β have not been assumed disjoint, we can have

$$G_\alpha \cap G_\beta = H_{\alpha\beta} = H_{\beta\alpha} \neq 1 \ .$$

If $H_{\alpha\beta} = 1$ for all $\alpha \neq \beta$, then G is called the *free product*, or the *ordinary free product* of the subgroups G_γ. In this paper, we assume that all the intersections $H_{\alpha\beta}$ coincide to form a single subgroup H. That is, we assume that $G_\alpha \cap G_\beta = H$, for all $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$. In this case, G is called the *free product of the subgroups G_γ with amalgamated subgroup H*. We denote G by

$$G = \star_{\gamma\in\Gamma}(G_\gamma; H_\gamma) \ ,$$

where H_γ is isomorphic with H for all $\gamma \in \Gamma$. If G is the free product of two subgroups A and B with amalgamated subgroup H, then we write $G = A \star_H B$.

4. **Main Results.** In 1972 C. Y. Tang published the following theorem [11, Lemma 2.5, p. 570]:

Theorem 1. Let $G = A \star_H B$, where H is any cyclic subgroup. If N is any subgroup of H normal in G, then $\Phi(A) \cap N$ and $\Phi(B) \cap N$ are contained in $\Phi(G)$.

Remark. In Theorem 1, since N is any subgroup of H normal in G, clearly, N can be replaced by $K(G, H)$.

Next, we state and prove Theorem 1 for infinitely many free factors.

<u>Theorem 2</u>. Let

$$G = \star_{\gamma \in \Gamma}(G_\gamma; H_\gamma)$$

be the free product of any collection of groups $\{G_\gamma\}_{\gamma\in\Gamma}$ with amalgamated subgroup H, where H is any cyclic subgroup. If N is any subgroup of H normal in G, then $\Phi(G_\gamma) \cap N$ is contained in $\Phi(G)$ for every $\gamma \in \Gamma$.

<u>Proof</u>. Let $x \in \Phi(G_\gamma) \cap N$, where γ is an arbitrary element of Γ. To show that $x \in \Phi(G)$, we need to prove that x is a nongenerator of G. Now, let S be a set consisting of elements of G such that $< S, x > = G$. But, since $< x >$ is a characteristic subgroup of N, and N is a normal subgroup of G, we deduce that $< x >$ is a normal subgroup of G. Hence, for every $g \in G_\gamma$ there exists an element $s_g \in < S >$ such that $g = s_g x^k$, for some integer k. Thus, $s_g = g(x^k)^{-1} \in G_\gamma$. This implies that $G_\gamma = < x, s_g : g \in G_\gamma >$. But, since $x \in \Phi(G_\gamma)$, we have $G_\gamma = < s_g : g \in G_\gamma > \ \leq \ < S >$, and thus $x \in < S >$. Therefore, $G = < S >$. Consequently, $x \in \Phi(G)$. This completes the proof.

Before stating and proving the exact analog of Theorem 2 for the lower near Frattini subgroups, we need the following three well-known theorems. Proofs of these three theorems can be found in [10, pp. 21–26].

Theorem 3. If H and K are any two subgroups of a group G such that $K \subseteq H$, then $|G : K| = |G : H||H : K|$.

Theorem 4. Let H and K be any two subgroups of a group G. If $|G : H|$ is finite, then $|K : H \cap K|$ is finite, and $|K : H \cap K| \leq |G : H|$. Equality holds if and only if $G = HK$.

Theorem 5. Let U and V be any two subsets, and let L be any subgroup of a group G. If $U \subseteq L$, then $UV \cap L = U(V \cap L)$.

Theorem 6. Let

$$G = \star_{\gamma \in \Gamma}(G_\gamma; H_\gamma)$$

be the free product of any collection of groups $\{G_\gamma\}_{\gamma \in \Gamma}$ with amalgamated subgroup H, where H is any cyclic subgroup. If N is any subgroup of H normal in G, then $\lambda(G_\gamma) \cap N$ is contained in $\lambda(G)$ for every $\gamma \in \Gamma$.

Proof. To prove $\lambda(G_\gamma) \cap N$ is contained in $\lambda(G)$, we need to show that every element of $\lambda(G_\gamma) \cap N$ is a non-near generator of G. Let $x \in \lambda(G_\gamma) \cap N$, and suppose that $S \subseteq G$ is such that $|G :< x, S > |$ is finite. We wish to show that $|G :< S > |$ is finite. Now, since $< x >$ is a characteristic subgroup of N, and N is a normal subgroup of G, we deduce that $< x >$ is a normal subgroup of G. Thus,

$$(1) \qquad |G :< x, S > | = |G :< x >< S > |$$

is finite, and by Theorem 4

$$|G \cap G_\gamma :< x >< S > \cap G_\gamma| = |G_\gamma :< x >< S > \cap G_\gamma|$$

is finite. Hence, by Theorem 5 $|G_\gamma :< x > (< S > \cap G_\gamma)|$ is finite. But, since x is a non-near generator of G_γ, and

$$|G_\gamma :< x > (< S > \cap G_\gamma)| = |G_\gamma :< x, < S > \cap G_\gamma > |$$

is finite, we deduce that $|G_\gamma :< S > \cap G_\gamma|$ is finite. Now, intersecting G_γ and $< S > \cap G_\gamma$ with $< x >$, and using Theorem 4 we have

$$(2) \qquad | < x >:< S > \cap < x > | = | < x >< S >:< S > |$$

which is finite. Finally, from (1), (2), and Theorem 3 it follows that

$$|G :< S > | = |G :< x >< S > || < x >< S >:< S > |$$

is finite. Therefore, $\lambda(G_\gamma) \cap N \leq \lambda(G)$. This completes the proof.

5. **Open Questions.**

Question 1. We have used the definition that $\Phi(G)$ is the set of all nongenerators of G, in the proof of Theorem 2. Can Theorem 2 be proven by using the definition that $\Phi(G)$ is the intersection of all maximal subgroups of G?

Question 2. Is the statement of Theorem 6 still true, if $\lambda(G)$ is replaced by $\mu(G)$?

References

1. M. K. Azarian, "On the Lower Near Frattini Subgroups and Nearly Splitting Groups," *Missouri Journal of Mathematical Sciences*, 2 (1990), 18–25.
2. M. K. Azarian, Near Frattini Subgroups of Amalgamated Free Products of Groups, Ph.D. dissertation, Saint Louis University, St. Louis, Missouri, 1985.
3. B. H. Neumann, "An Essay on Free Products of Groups With Amalgamations," *Philos. Trans. Roy. Soc. London Ser.*, A 246 (1954), 503–554.
4. H. Neumann, "Generalized Free Products with Amalgamated Subgroups I," *Amer. J. Math.*, 70 (1948), 590–625.

5. H. Neumann, "Generalized Free Products with Amalgamated Subgroups II," *Amer. J. Math.*, 71 (1949), 491–540.

6. W. Magnus, A. Karrass, and D. Solitar, Combinatorial Group Theory : Presentations of Groups in Terms of Generators and Relations, Second Revised Edition, Dover Publications, New York, New York, 1976.

7. J. B. Riles, "The Near Frattini Subgroups of Infinite Groups," *Journal of Algebra*, 12 (1969), 155–171.

8. O. Schreier, "Die Untergruppen der freien Gruppen," *Abh. Math. Sem. Hamburg*, 5 (1927), 161–183.

9. W. R. Scott, Group Theory, Dover Publications, New York, New York, 1987.

10. M. Suzuki, Group Theory I, Springer-Verlag, New York, New York, 1982.

11. C. Y. Tang, "On the Frattini Subgroups of Generalized Free Products With Cyclic Amalgamations," *Canad. Math. Bull.*, 15 (1972), 569–573.