NOETHERIAN INTEGRALLY CLOSED DUO RINGS ARE KRULL RINGS

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Introduction. A ring R with unity is called a duo ring if every right ideal of R is a left ideal and conversely. This is equivalent to the condition that $aR = Ra$ for all $a \in R$. When R is an integral domain, R has a (left and right) division ring D of quotients, and $xRx^{-1} = R$ for all nonzero $x \in D$. In this sense R is an invariant subring of D. In this note we characterize Noetherian, integrally closed duo domains [1] in terms of valuation rings exactly as commutative Krull domains are characterized. We continue in this manner to characterize duo domains which are unique factorization domains [3] as Krull domains in which the height 1 prime ideals are principal. We then show that when R has an Abelian group of divisibility, then R is integrally closed if and only if R is an intersection of valuation rings. Duo domains which are Prüfer domains with Abelian group of divisibility are characterized exactly as commutative Prüfer domains.

In what follows, R denotes a duo ring which is an integral domain, D its division ring of quotients. Prime ideals in duo rings have the same characterization as in the commutative case [10]. Thus P is a prime ideal of R if and only if $S = R\backslash P$ is a multiplicative system in R.

A subset T of R is called invariant if $aT = Ta$ for all $a \in R$. This is clearly equivalent to the condition that $x^{-1}Tx = T$ for all $x \in D^* = D\backslash\{0\}$. Invariance is significant when studying localization in duo rings. The following was show in [6].

Theorem 1. If S is a multiplicative system in R then R_S is a duo ring if and only if S is invariant.

Let U denote the multiplicative group of units of R. It was shown in [3] that U is a normal subgroup of D^*. As in the commutative case, D^*/U is called the group of divisibility of R.

When R is Noetherian and integrally closed, it is shown in [1] that for all $a, b \in R$, $aRbR = bRaR$. Since $aRbR = abR$, this is equivalent to the condition that for all $a, b \in R$, $ab = bau$, for some $u \in U$. We can now state the following.

Theorem 2. In a duo domain R, the following are equivalent.

(1) The multiplicative semigroup of ideals of R is Abelian.

(2) The multiplicative semigroup of principal ideals of R is Abelian.

(3) The group of divisibility of R, D^*/U, is Abelian.

(4) Every ideal of R is invariant.

(5) Every principal ideal of R is invariant.

Proof. Since every ideal is a sum of principal ideals (1) $\Leftrightarrow$ (2), and (4) $\Leftrightarrow$ (5). (2) $\Leftrightarrow$ (3) follows from the remarks preceding

Theorem 2. (5) $\Leftrightarrow$ (3) is easy to show.

We say that R satisfies property (A) if it satisfies the equivalent conditions of Theorem 2.

Corollary 3 [1]. If R satisfies property (A) then every overring T of R $(R \subseteq T \subseteq D)$ is a duo ring. In particular R_S is a duo ring for every multiplicative system S in R.

When R is Noetherian and integrally closed, in addition to satisfying property (A), the following was shown in [1]: If $\{P_\lambda\}$ denotes the collection of height 1 prime ideals of R then

(a) $R = \bigcap_\lambda R_{P_\lambda}$.

(b) $P_\lambda R_{P_\lambda} = p_\lambda R_{P_\lambda}$ for some $p_\lambda \in P_\lambda$.

(c) The only nonzero ideals of $p_\lambda R_{P_\lambda}$ are the ideals of $p_\lambda^n R_{P_\lambda}$.

If we define $v_\lambda : R_{P_\lambda} \to (Z, +) \cup \{\infty\}$ by $v_\lambda(x) = n$ where n is the unique integer such that $x \in p_\lambda^n R_{P_\lambda} - p_\lambda^{n+1} R_{P_\lambda}$ then v_λ satisfies the following: (I) $v_\lambda(0) = \infty$, (II) $v_\lambda(xy) = v_\lambda(x+y)$ for all x, $y \in R_{P_\lambda}$, (III) $v_\lambda(x+y) \geq \min(v(x), v(y))$ for x, $y \in R_{P_\lambda}$.

Non-commutative valuations and noncommutative valuation rings were studied in [7] and [8].

Let $(G, \bullet)$ be a totally ordered group. Let ∞ be a symbol not in G. Let $v : D \to G \cup \{\infty\}$. We define $x * \infty = \infty = \infty * x$ for any $x \in G$ and $x < \infty$ for all $x \in G$.

Definition 4 [8]. v is called a valuation on D if

(1) $v(0) = \infty$.

(2) $v(xy) = v(x)v(y)$ for any x , $y \in D$.

(3) $v(x+y) \geq \min(v(x) , v(y))$.

(Here the operation in G is denoted by juxtaposition.)

Theorem 5 [12]. Let $v : R \to G \cup \{\infty\}$ satisfy (1), (2), (3) above. There is a unique extension v' of v to a valuation on D .

Proof. For $x = b^{-1}a \in D^*$, define $v'(x) = v(b)^{-1}v(a)$. Clearly (1) is satisfied. To see (2), let $x = b^{-1}a$, $y = d^{-1}c$. If $x = 0$ or $y = 0$, (2) is clear. Suppose x , $y \neq 0$. Then $v'(xy) = v'(b^{-1}ad^{-1}c)$. Now $ad = d_1a$ for some $d_1 \in R$, so $ad^{-1} = d_1^{-1}a$, so

$$
\begin{aligned}
v'(xy) = v'(b^{-1}d_1^{-1}ac) &= v'((d_1b)^{-1}(ac)) = v(d_1b)^{-1}v(ac) \\
&= v(b)^{-1}v(d_1)^{-1}v(a)v(c) = [v(b)^{-1}v(a)][v(d)^{-1}v(c)] \\
&= v'(x)v'(y) \ ,
\end{aligned}
$$

and (2) is proved.

To see (3), let $x = b^{-1}a$, $y = d^{-1}c$. If $x+y = 0$ there is nothing to show. So suppose $x+y \neq 0$. Then $x+y = (b_1d)^{-1}(d_1a + b_1c)$, where $d_1b = b_1d$.

$$
\begin{aligned}
v'(x+y) &= v(b_1d)^{-1}v(d_1a + b_1c) \\
&\geq v(b_1d)^{-1}\min(v(d_1a) , v(b_1c)) \\
&= \min(v(b_1d)^{-1}v(d_1a) , v(b_1d)^{-1}v(b_1d)) \ .
\end{aligned}
$$

Now $d_1 b = b_1 d$, so $v(d_1 b) = v(b_1 d)$, and hence

$$v(b_1 d)^{-1} = v(b)^{-1} v(d_1)^{-1} = v(d)^{-1} v(b_1)^{-1} .$$

So $v'(x+y) \geq \min(v'(b^{-1}a), v'(d^{-1}c))$, that is,

$$v'(x+y) \geq \min(v'(x), v'(y)) .$$

Uniqueness is clear, since extension w of v satisfies $w(b^{-1}a) = w(b^{-1})w(a) = w(b^{-1})v(a)v(b)^{-1}v(a)$. For $w(b^{-1}b) = w(1) = e$, the identity of G, and $w(b^{-1}) = w(b)^{-1} = v(b)^{-1}$.

We observe that the proof of Theorem 5 did not require that R satisfy property (A).

It follows from Theorem 5 that each v_λ can be extended to a valuation v_λ on D, and that each valuation ring $R_{v_\lambda} = R_{P_\lambda}$ and is rank one, discrete.

So when R is Noetherian and integrally closed, the family $\{P_\lambda\}$ of height 1 prime ideals induces a family F of valuations on D which has the following properties.

(i) $R = \bigcap\{R_v : v \in F\}$.

(ii) Each $v \in F$ has rank one and is discrete.

(iii) F is of finite character, i.e., for $a \in R$, $v(a) \neq 0$ for only finitely many $v \in F$.

(iv) $R_v = R_{P(v)}$, where $P(v) = \{x \in R : v(x) > 0\}$.

So a Noetherian, integrally closed duo domain satisfies the classical definition of a commutative Krull domain [12].

Accordingly, we shall call a duo domain R a Krull domain if there is a family F of valuations on the division ring D of quotients of R satisfying (i), (ii), (iii), (iv) above. Clearly, if R is a Krull domain then R satisfies property (A) since each R_v has an Abelian group of divisibility. Thus every multiplicative system S in a Krull domain is invariant.

We now state the following. The proof is like the commutative case.

<u>*Theorem* 6</u>. Let R be a Krull domain with family F of essential valuations. Let S be a multiplicative system in R. Let

$$H = \{v \in F : P(v) \cap S = \emptyset\} .$$

Then R_S is a Krull domain with H as family of essential valuations.

Duo rings which are unique factorization domains (UFD) were first studied in [3, pg. 85]. We state definitions from [3].

<u>*Definition* 7</u>. Let R be a duo domain.

(1) $x \in R$ is a prime if $xR = Rx$ is a prime ideal of R.

(2) x, $y \in R$ are associates if $Rx = Ry$. (Then $xR = yR$.)

(3) For a, $b \in R$, a divides b $(a|b)$ if $bR \subseteq aR$.

(4) $x \in R$ is irreducible if $x = yz$ implies either y or z is a unit.

(5) R is called a UFD if the following are satisfied.

<u>*UF*1</u>. Every nonzero element is a product of a finite number of irreducible elements.

UF2. If $a_1 a_2 \cdots a_s = b_1 b_2 \cdots b_t$ where a_1, a_2, $\ldots$, a_s, b_1, b_2, $\ldots$, b_t are nontrivial irreducibles, then $s = t$ and for a suitable ordering of the subscripts, a_i is an associate of b_i.

It follows from [3] that if R is UFD then irreducible elements are prime.

Proposition 8. If R is UFD and if x, $y \in R$ are nonassociate prime elements then $(xR)(yR) = (yR)(xR)$, i.e., $xyR = yxR$.

Proof. Since R is duo, $xy = yx'$ for some $x' \in R$. Then $y^{-1}xy = x'$, and $y^{-1}xyR = y^{-1}(xR)y = x'R$. Then $x'R$ is the conjugate of a prime ideal and so is a prime ideal. So $x' \in R$ is prime and so by UF2, x' is an associate of x, say $x' = xt$ where t is a unit. Then $xyR = yxtR = yxR$.

The above proposition shows that if R is UFD then R satisfies property (A), by UF1.

Let $\{x_\lambda\}$ denote a collection of primes of R such that

(a) x_α is not an associate of x_β for $\alpha \neq \beta$.

(b) If $r \in R$ is not a unit, then $x_\lambda | r$ for some λ.

Let $P_\lambda = x_\lambda R$. Then as in [12], each $x_\lambda R_{P_\lambda}$ is a rank one discrete valuation ring, and it is like the commutative case to show that each P_λ induces a family F of valuations satisfying (i), (ii), (iii), (iv).

Theorem 9. Let R be a duo domain with D as a division ring of quotients. D is UFD if and only if

(1) R is a Krull domain.

(2) Every height 1 prime ideal is principal.

The proof is like the commutative case [12].

As in the commutative case, an element $x \in D$ is integral over R if there exist $a_0, \ldots, a_{n-1} \in R$ such that $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$. This definition is symmetric:

$$\begin{aligned} & x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\ & \quad = x^n + x^{n-1}(x^{n-1})^{-1}a_{n-1}x^{n-1} + \cdots + x(x^{-1}a_1 x) + a_0 , \end{aligned}$$

and $(x^j)^{-1}a_j x^j \in R$, since R is duo, for $j = 1, \ldots, n-1$. It was shown in [7] that (noncommutative) valuation rings are integrally closed. As was shown earlier, when R is Noetherian and integrally closed, then R is an intersection of valuation rings. It is well known [12] that an integrally closed commutative integral domain is an intersection of valuation rings. We have been able to prove the following:

Theorem 10. Let R be a duo ring satisfying property (A). Then R is integrally closed if and only if R is an intersection of valuation rings.

The proof of Theorem 10 will follow from Lemmas 11 and 12.

Lemma 11 [12]. Let I be an ideal in R. Then for any $x \in D$, at least one of $R[x]I$, $R[x^{-1}]I$ is a proper ideal of $R[x]$, $R[x^{-1}]$ respectively.

Proof. The proof is like the commutative case, using property (A) in R and the invariance of R. See [12, pg. 11].

Lemma 12 [12]. Let P be a prime ideal of R. There exists a valuation overring V of R $(R \subseteq V \subseteq D)$ with maximal ideal $\mathcal{P}$ such that $\mathcal{P} \cap R = P$.

Proof. We may assume $R = R_P$, so that P is the unique maximal ideal of R. Let $\mathcal{S} = \{T : T \text{ is the overring of } R,\ PT \neq T\}$, $\mathcal{S} \neq \emptyset$. A Zorn's lemma argument shows $\mathcal{S}$ has a maximal element, say T. The maximality of P in R_P gives that PT is maximal. If not, then $PT < M$ for some maximal ideal M of T and then $T < T_M$, $T_M \in \mathcal{S}$, a contradiction. Clearly $PT \cap R = P$.

Applying Lemma 11 to $R = T$, $I = PT$, as in the commutative case [12, pg. 12] we get that for $x \in D$, either $x \in T$ or $x^{-1} \in T$, and T is a valuation ring, since T is duo.

So if $b \in D \backslash R$, then consider $R[b^{-1}]$ (see [5, pg. 38]). If $b^{-1}\left(\sum_{i=0}^{n} a_i (b^{-1})^i\right) = 1$, $a_0, \ldots, a_n \in R$, then b is integral over R, which is not the case if R is integrally closed. So b^{-1} is not a unit in $R[b^{-1}]$. Then b^{-1} belongs to a maximal, hence prime ideal of $R[b^{-1}]$. Applying Lemma 13, we get a valuation ring V, $R \subseteq R[b^{-1}] \subseteq V \subseteq D$ with b^{-1} a nonunit in V. Then $b \notin V$. So if R is integrally closed it is the intersection of valuation rings.

Since (noncommutative) valuation rings are integrally closed [7], this completes the proof of Theorem 10.

Strong use of property (A) was made in the proof. The author believes the following general case to be true but has been unable to prove it: For any duo domain R, R is integrally closed if and only if R is an intersection of valuation rings.

As in the commutative case, R is called a Dedekind domain if R is a Krull domain in which each height 1 prime ideal is maximal, i.e., $P(v)$ is maximal for each $v \in F$. Since R satisfies property (A), all the equivalent conditions for a commutative integral domain to be a Dedekind domain are satisfied in R. So R is Dedekind if and only if every nonzero ideal of R is invertible [5], [4].

The following is found in [9]: A (not necessarily commutative) ring T is called a left AM-ring (respectively right AM-ring) is for all ideal $A < B$ of T (here $<$ denotes proper containment) there is an ideal C such that $A = BC$ (respectively $A = CB$). T is an AM-ring if it is both a left and right AM-ring. (In the commutative case, T is sometimes called a multiplication ring [4, pg. 209].)

Theorem 3.1 of [9] shows that if R is an AM-ring then R satisfies property (A). As in the commutative case, we have the following [4, pg. 210].

Theorem 13. If R is an AM-ring then R is a Dedekind domain.

Proof. Let A be a nonzero ideal of R. If $A = Ra$ then A

is invertible. If A is not principal, then there is nonzero $a \in A$ such that $Ra < A$. Then $Ra = AC$ for some ideal C, and A is invertible, with $A^{-1} = CRa^{-1}$. Thus R is Dedekind.

We call R a Prüfer domain if R_M is a (noncommutative) valuation ring for each maximal ideal M of R. Noncommutative valuation rings [7] are examples of duo rings which are Prüfer domains. Ten equivalent conditions for a commutative integral domain to be a Prüfer domain are given in Theorem 6.6 of [4]. Theorem 6.6 of [4] also holds when R is a duo domain which satisfies property (A). The proof is the same as the commutative case because of property (A).

An example of a noncommutative valuation ring in which R_P is not a valuation ring for every prime P was given in [7]. Thus there exist noncommutative valuation rings which do not satisfy the ten equivalent conditions of Theorem 6.6 of [4].

When R is commutative, it is well known that if R is a Krull domain, then so is the polynomial ring $R[X]$. Commutativity appears to be necessary, for even when R is duo and satisfies property (A), $R[X]$ is in general not a duo ring. For example, if D is a noncommutative division ring, then D satisfies property (A) but $D[X]$ is not a duo ring.

An example of a duo domain which is a Krull domain is the power series ring $D[[X]]$ where D is any division ring and X is a central indeterminate. $D[[X]]$ is a noncommutative rank one discrete valuation ring with $XD[[X]]$ as the unique maximal ideal.

So $D[[X]]$ is a noncommutative duo UFD, and a Dedekind domain.

Other examples of integrally closed duo rings which satisfy property (A) may be constructed from results in [2]. If D is a 2-finite division algebra over its center k and satisfies the equivalent conditions of Theorem 1 of [2], then any real valued valuation v on k may be extended to a valuation w on D. So the valuation w has Abelian value group, and R_w, the valuation ring of w, satisfies property (A) and is integrally closed. If $\{R_{w_\lambda}\}$ is any collection of such valuation rings, then $R = \cap_\lambda R_{w_\lambda}$ is a duo ring which satisfies property (A) and is integrally closed.

References

1. H. H. Brungs, "Three Questions on Duo Rings," *Pacific J. Math.*, 58(1975) 345–349.
2. P. M. Cohn, "On Extending Valuations in Division Algebras," *Studia Sci. Math. Hung.*, 16(1981) 65–70.
3. E. H. Feller, "Properties of Primary Non-Commutative Rings," *Trans. Amer. Math. Soc.*, 89(1958) 79–91.
4. M. D. Larsen and P. J. McCarthy, *Multiplicative Theory of Ideals*, Academic Press, New York, 1971.
5. M. Nagata, *Local Rings*, Interscience, New York, 1962.
6. E. M. Pirtle, "Localization in Duo Rings," *Pub. Math. Debrecen*, 31 (1984) 47–52.
7. E. M. Pirtle, "Noncommutative Valuation Rings," *Pub. de l'Institut Mathematique*, 39 (1986) 83–87.
8. E. F. G. Schilling, "Noncommutative Valuations," *Bull. Amer. Math. Soc.*, 51 (1945) 297–304.
9. P. F. Smith, "On Non-Commutative AM-Rings," *Houston Journal of Math.*, 11 (1985) 405–422.
10. G. Thierrin, "On Duo Rings," *Can. Math. Bull.*, 3 (1960) 167–

172.

11. O. Zariski and P. Samuel, *Commutative Algebra*, New York, Vol. I, 1958.

12. O. Zariski and P. Samuel, *Commutative Algebra*, New York, Vol. II, 1962.