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The Dirichlet problem for a prescribed mean curvature equation

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Abstract. We study a prescribed mean curvature problem where we seek a surface whose mean curvature vector coincides with the normal component of a given vector field. We prove that the problem has a solution near a graphical minimal surface if the prescribed vector field is sufficiently small in a dimensionally sharp Sobolev norm.

1. Introduction

In this paper, we consider the following prescribed mean curvature problem with the Dirichlet condition,

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = H(x, u(x), \nabla u(x)) \quad \text{in } \Omega, \\ u = \phi \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$. The function $H(x,t,z) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is given and we seek a solution u satisfying (1). Since the left hand side of (1) is the mean curvature of the graph of u, (1) is a prescribed mean curvature equation whose prescription depends on the location of the graph as well as the slope of the tangent space.

Prescribed mean curvature problems in a wide variety of formulation have been studied by numerous researchers. In the most classical case of $H = H(x)$, (1) has a solution if H and ϕ have suitable regularity and the mean curvature of $\partial\Omega$ satisfies a certain geometric condition (see [3, 4, 6, 7, 8, 11], for example). Giusti [5] determined a necessary and sufficient condition that a prescribed mean curvature problem without boundary conditions has solutions. In the case of $H = H(x,t)$, Gethardt [2] constructed $H^{1,1}$ solutions, and Miranda [10] constructed BV solutions. In those papers, assumptions of the boundedness $|H| < \infty$ and the monotonicity $\frac{\partial H}{\partial t} \geq 0$ play an important role. If $|H| < \Gamma$ where Γ is determined by Ω, there exist solutions of (1), and the uniqueness of solutions is guaranteed by the monotonicity, that is, $\frac{\partial H}{\partial t} \geq 0$. Under the

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assumptions of boundedness, monotonicity and the convexity of Ω, Bergner [1] solved the Dirichlet problem in the case of $H = H(x, u, \nu(\nabla u))$ using the Leray-Schauder fixed point theorem. Here, ν is the unit normal vector of u, that is, $\nu(z) = \frac{1}{\sqrt{1+|z|^2}}(z, -1)$. For the same problem as [1], Marquardt [9] gave a condition on $\partial\Omega$ depending on H which guarantees the existence of solutions even for a non-convex domain Ω.

The motivation of the present paper comes from a singular perturbation problem studied in [12], where one considers the following problem on a domain $\tilde{\Omega} \subset \mathbb{R}^{n+1}$,

$$-\varepsilon\Delta\phi_\varepsilon + \frac{W'(\phi_\varepsilon)}{\varepsilon} = \varepsilon\nabla\phi_\varepsilon \cdot f_\varepsilon. \tag{2}$$

Here, W is a double-well potential, for example $W(\phi) = (1-\phi^2)^2$ and $\{f_\varepsilon\}_{\varepsilon>0}$ are given vector fields uniformly bounded in the Sobolev norm of $W^{1,p}(\tilde{\Omega})$, $p > \frac{n+1}{2}$. In [12], we proved under a natural assumption

$$\int_{\tilde{\Omega}} \left(\frac{\varepsilon|\nabla\phi_\varepsilon|^2}{2} + \frac{W(\phi_\varepsilon)}{\varepsilon} \right) dx + \|f_\varepsilon\|_{W^{1,p}(\tilde{\Omega})} \le C \tag{3}$$

that the interface $\{\phi_\varepsilon = 0\}$ converges locally in the Hausdorff distance to a surface whose mean curvature H is given by $f \cdot \nu$ as $\varepsilon \to 0$. Here, f is the weak $W^{1,p}$ limit of f_ε. If the surface is represented locally as a graph of a function u over a domain $\Omega \subset \mathbb{R}^n$, the corresponding relation between the mean curvature and the vector field is expressed as

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = \nu(\nabla u(x)) \cdot f(x, u(x)) \qquad \text{in } \Omega, \tag{4}$$

where $f \in W^{1,p}(\Omega \times \mathbb{R}; \mathbb{R}^{n+1})$ with $p > \frac{n+1}{2}$. Note that f is not bounded in L^∞ in general, unlike the cases studied in [1, 9]. In this paper, we establish the well-posedness of the perturbative problem including (4) which has a $W^{1,p}$ norm control on the right-hand side of the equation. The following theorem is the main result of this paper.

THEOREM 1. *Let Ω be a $C^{1,1}$ bounded domain in $\mathbb{R}^n$ and fix constants $\varepsilon > 0$, $\frac{n+1}{2} < p < n+1$ and $q = \frac{np}{n+1-p}$. Suppose $h \in W^{2,\infty}(\Omega)$ satisfies the minimal surface equation, that is,*

$$\operatorname{div}\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) = 0. \tag{5}$$

Then there exists a constant $\delta_1 > 0$ *which depends only on* n, p, Ω, $\|h\|_{W^{2,\infty}(\Omega)}$, *and* ε *with the following property. Suppose* $G \in W^{1,p}(\Omega \times \mathbb{R})$ *and* $\phi \in W^{2,q}(\Omega)$ *satisfy*

$$\|G\|_{W^{1,p}(\Omega\times\mathbb{R})} + \|\phi\|_{W^{2,q}(\Omega)} \le \delta_1, \tag{6}$$

and a measurable function $H(x,t,z) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *is such that* $H(x,\cdot,\cdot)$ *is a continuous function for* a.e. $x \in \Omega$, *and for all* $(t,z) \in \mathbb{R} \times \mathbb{R}^n$,

$$|H(x,t,z)| \le |G(x,t)| \qquad \textit{for } \text{a.e. } x \in \Omega. \tag{7}$$

Then, there exists a function $u \in W^{2,q}(\Omega)$ *such that* $u - h - \phi \in W_0^{1,q}(\Omega)$ *and*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = H(x,u(x),\nabla u(x)) \qquad \textit{in } \Omega, \tag{8}$$

$$\|u-h\|_{W^{2,q}(\Omega)} < \varepsilon. \tag{9}$$

The claim proves that there exists a solution of (1) in a neighbourhood of any minimal surface if H and ϕ are sufficiently small in these norms. In particular, if we take $H(x,t,z) = v(z)\cdot f(x,t)$ and $G(x,t) = |f(x,t)|$, where $\|f\|_{W^{1,p}(\Omega\times\mathbb{R})}$ is sufficiently small, above conditions on G and H in Theorem 1 are satisfied and we can guarantee the existence of a solution for (1) nearby the given minimal surface (see Corollary 1). The method of proof is as follows. We prove that the linearized problem of (1) has a unique solution in $W^{2,q}(\Omega)$ and the norm of this solution is controlled by G and ϕ. When (6) is satisfied, there exist a suitable function space $\mathscr{A}$ and a mapping $T : \mathscr{A} \to \mathscr{A}$, and a fixed point of T is a solution of (8) with $u - h - \phi \in W_0^{1,q}(\Omega)$. We show that T satisfies assumptions of the Schauder fixed point theorem, and Theorem 1 follows.

2. Proof of Theorem 1

Throughout the paper, Ω is a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary $\partial\Omega$. We define functions $A_{ij} : \mathbb{R}^n \to \mathbb{R}$ $(i,j = 1,\dots,n)$ as

$$A_{ij}(z) := \frac{1}{\sqrt{1+|z|^2}}\left(\delta_{ij} - \frac{z_i z_j}{1+|z|^2}\right)$$

and the operator

$$L[z](u) := A_{ij}(z)D_{ij}u(x) \qquad \text{for any } u \in W^{2,1}(\Omega),$$

where we omit the summation over $i, j = 1, \dots, n$. By the Cauchy–Schwarz inequality, for any $\xi \in \mathbb{R}^n$,

$$\begin{aligned} A_{ij}(z)\xi_i\xi_j &= \frac{1}{\sqrt{1+|z|^2}}\left(\delta_{ij} - \frac{z_i z_j}{1+|z|^2}\right)\xi_i\xi_j \\ &= \frac{1}{\sqrt{1+|z|^2}}\left[|\xi|^2 - \left(\frac{z_i}{\sqrt{1+|z|^2}}\xi_i\right)^2\right] \\ &\geq \frac{1}{\sqrt{1+|z|^2}}\left[|\xi|^2 - \left(\frac{|z|^2}{1+|z|^2}\right)|\xi|^2\right] \\ &= \frac{1}{(1+|z|^2)^{3/2}}|\xi|^2. \end{aligned} \tag{10}$$

Hence, as is well-known, the operator $L[z]$ is elliptic.

Theorem 2. *Suppose* $v \in C^{1,\alpha}(\bar{\Omega})$ *with* $0 < \alpha < 1$, $B = (B_1, \dots, B_n) \in L^\infty(\Omega; \mathbb{R}^n)$ *with* $\|B_i\|_{L^\infty(\Omega)} \leq K$ *for all* $i \in \{1, \dots, n\}$, $f \in L^q(\Omega)$ *and* $\phi \in W^{2,q}(\Omega)$ *with* $q > n$. *Then there exists a unique function* $u \in W^{2,q}(\Omega)$ *such that*

$$\begin{cases} L[\nabla v](u) + B \cdot \nabla u = f & \text{in } \Omega, \\ u - \phi \in W_0^{1,q}(\Omega). \end{cases} \tag{11}$$

Moreover, there exists a constant c_0 *which depends only on* n, q, Ω, K, *and* $\|v\|_{C^{1,\alpha}(\bar{\Omega})}$ *such that*

$$\|u\|_{W^{2,q}(\Omega)} \leq c_0(\|f\|_{L^q(\Omega)} + \|\phi\|_{W^{2,q}(\Omega)}). \tag{12}$$

Proof. By (10), for any $\xi \in \mathbb{R}^n$,

$$A_{ij}(\nabla v)\xi_i\xi_j \geq \frac{1}{(1+\|v\|^2_{C^{1,\alpha}(\bar{\Omega})})^{3/2}}|\xi|^2 =: \lambda|\xi|^2, \tag{13}$$

where the constant λ depends only on $\|v\|_{C^{1,\alpha}(\bar{\Omega})}$. Since each A_{ij} is a smooth function of ∇v, there exists a constant Λ which depends only on $\|v\|_{C^{1,\alpha}(\bar{\Omega})}$ such that

$$\|A_{ij}(\nabla v)\|_{C^{0,\alpha}(\bar{\Omega})} \leq \Lambda \qquad \text{for all } i, j \in \{1, \dots, n\}. \tag{14}$$

By (13) and (14), there exists a unique solution $u \in W^{2,q}(\Omega)$ satisfying (11) (cf. [4, Theorem 9.15]). Using [4, Theorem 9.13], we can know that there exists a constant c_1 which depends only on n, q, Ω, λ, K, and Λ such that

$$\|u\|_{W^{2,q}(\Omega)} \le c_1(\|u\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)} + \|\phi\|_{W^{2,q}(\Omega)}). \tag{15}$$

Using the Aleksandrov maximum principle [4, Theorem 9.1], we can know that there exists a constant c_2 which depends only on n, Ω, K, and λ such that

$$\begin{aligned}\|u\|_{L^\infty(\Omega)} &\le \sup_{x\in\partial\Omega}|u| + c_2\|f\|_{L^n(\Omega)}\\ &= \sup_{x\in\partial\Omega}|\phi| + c_2\|f\|_{L^n(\Omega)}.\end{aligned} \tag{16}$$

By the Hölder and Sobolev inequalities, $\phi \in C(\bar{\Omega})$ and

$$\begin{aligned}\|u\|_{L^q(\Omega)} &\le c\|u\|_{L^\infty(\Omega)}\\ &\le c\left(\sup_{x\in\partial\Omega}|\phi| + \|f\|_{L^n(\Omega)}\right)\\ &\le c(\|\phi\|_{C(\bar{\Omega})} + \|f\|_{L^n(\Omega)})\\ &\le c_3(\|f\|_{L^q(\Omega)} + \|\phi\|_{W^{2,q}(\Omega)}),\end{aligned} \tag{17}$$

where c_3 depends only on n, q, and Ω. By (15) and (17), there exists a constant c_0 which depends only on n, q, Ω, λ, K, and Λ such that

$$\|u\|_{W^{2,q}(\Omega)} \le c_0(\|f\|_{L^q(\Omega)} + \|\phi\|_{W^{2,q}(\Omega)}). \tag{18}$$

Thus this theorem follows. □

To proceed, we need the following theorem (cf. [13, Theorem 5.12.4]).

Theorem 3. *Let μ be a positive Radon measure on $\mathbb{R}^{n+1}$ satisfying*

$$K(\mu) := \sup_{B_r(x)\subset\mathbb{R}^{n+1}} \frac{1}{r^n}\mu(B_r(x)) < \infty.$$

Then there exists a constant c_4 which depends only on n such that

$$\left|\int_{\mathbb{R}^{n+1}} \phi\, d\mu\right| \le c_4 K(\mu)\int_{\mathbb{R}^{n+1}} |\nabla\phi|\, d\mathscr{L}^{n+1}$$

for all $\phi \in C_c^1(\mathbb{R}^{n+1})$.

Lemma 1. *Suppose $v \in W^{1,\infty}(\Omega)$ with $\|v\|_{W^{1,\infty}(\Omega)} \le V$ and $G \in W^{1,p}(\Omega\times\mathbb{R})$ with $\frac{n+1}{2} < p < n+1$. Let $q = \frac{np}{n+1-p}$. Then there exists a constant c_5 which depends only on n, p, Ω, and V such that*

$$\|G(\cdot, v(\cdot))\|_{L^q(\Omega)} \le c_5\|G\|_{W^{1,p}(\Omega\times\mathbb{R})}. \tag{19}$$

Proof. Define

$$\Gamma := \{(x, v(x)) \in \Omega \times \mathbb{R}\}.$$

A set $B_r^n(x)$ is the open ball with center x and radius r in $\mathbb{R}^n$. In the following, $\mathscr{H}^n$ denotes the n-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$ and $\mathscr{H}^n \llcorner_\Gamma$ is a Radon measure defined by

$$\mathscr{H}^n \llcorner_\Gamma(A) := \mathscr{H}^n(A \cap \Gamma) \qquad \text{for all } A \subset \mathbb{R}^{n+1}.$$

Then the support of $\mathscr{H}^n \llcorner_\Gamma$ satisfies in particular spt $\mathscr{H}^n \llcorner_\Gamma \subset \Omega \times (-2V, 2V)$. For any $B_r^{n+1}((x_0, x_0')) \subset \mathbb{R}^{n+1}$ with $(x_0, x_0') \in \mathbb{R}^n \times \mathbb{R}$,

$$\frac{1}{r^n} \mathscr{H}^n \llcorner_\Gamma (B_r^{n+1}((x_0, x_0'))) \le \frac{1}{r^n} \int_{B_r^n(x_0) \cap \Omega} \sqrt{1 + |\nabla v|^2}\, d\mathscr{L}^n \le (1 + V)\omega_n, \tag{20}$$

where ω_n is the volume of n-dimensional unit open ball. Using the standard Extension Theorem, we can know that there exists a function $\tilde{G} \in W^{1,p}(\mathbb{R}^{n+1})$ such that $\tilde{G} = G$ in $\Omega \times (-2V, 2V)$ and

$$\|\tilde{G}\|_{W^{1,p}(\mathbb{R}^{n+1})} \le c_6 \|G\|_{W^{1,p}(\Omega \times (-2V, 2V))}, \tag{21}$$

where c_6 depends only on n, p, Ω, and V. By Theorem 3 and smoothly approximating $\tilde{G}$,

$$\begin{aligned}
\int_\Omega |G(x, v(x))|^q dx &\le \int_\Omega |\tilde{G}(x, v(x))|^q \sqrt{1 + |\nabla v|^2}\, dx \\
&= \int_\Gamma |\tilde{G}(x, x_{n+1})|^q d\mathscr{H}^n \\
&\le c(n, V) \int_{\mathbb{R}^{n+1}} |\nabla \tilde{G}|\, |\tilde{G}|^{q-1} d\mathscr{L}^{n+1} \\
&\le c(n, p, V) \|\nabla \tilde{G}\|_{L^p(\mathbb{R}^{n+1})} \|\tilde{G}\|_{W^{1,p}(\mathbb{R}^{n+1})}^{q-1} \\
&\le c(n, p, V) c_6 \|G\|_{W^{1,p}(\Omega \times (-2V, 2V))}^q \\
&\le c(n, p, V) c_6 \|G\|_{W^{1,p}(\Omega \times \mathbb{R})}^q.
\end{aligned} \tag{22}$$

This lemma follows. □

We write the Schauder fixed point theorem needed later ([4, Corollary 11.2]).

Theorem 4. *Let $\mathscr{G}$ be a closed convex set in Banach space $\mathscr{B}$ and let T be a continuous mapping of $\mathscr{G}$ into itself such that the image $T(\mathscr{G})$ is precompact. Then T has a fixed point.*

We first prove Theorem 1 in the case that $h = 0$.

THEOREM 5. *Assume that* $G \in W^{1,p}(\Omega \times \mathbb{R})$ *with* $\frac{n+1}{2} < p < n+1$ *and* $\phi \in W^{2,q}(\Omega)$ *with* $q = \frac{np}{n+1-p}$. *Then there exists a constant* $\delta_2 > 0$ *which depends only on* n, p, *and* Ω *such that, if*

$$\|G\|_{W^{1,p}(\Omega \times \mathbb{R})} + \|\phi\|_{W^{2,q}(\Omega)} \le \delta_2, \tag{23}$$

then, for any measurable function $H(x,t,z) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *such that* $H(x,\cdot,\cdot)$ *is a continuous function for* a.e. $x \in \Omega$ *and*

$$|H(x,t,z)| \le |G(x,t)| \qquad \textit{for a.e. } x \in \Omega, \textit{ any } (t,z) \in \mathbb{R} \times \mathbb{R}^n, \tag{24}$$

there exists a function $u \in W^{2,q}(\Omega)$ *such that* $u - \phi \in W_0^{1,q}(\Omega)$ *and*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = H(x,u(x),\nabla u(x)) \qquad \textit{in } \Omega. \tag{25}$$

PROOF. Define

$$\mathscr{A} := \{v \in C^{1,1/2-n/2q}(\bar{\Omega}); \|v\|_{C^{1,1/2-n/2q}(\bar{\Omega})} \le 1\}. \tag{26}$$

The set $\mathscr{A}$ is a closed convex set in Banach space $C^{1,1/2-n/2q}(\bar{\Omega})$. By (24) and Lemma 1, $H(\cdot,v(\cdot),\nabla v(\cdot)) \in L^q(\Omega)$ for any $v \in \mathscr{A}$. Using Theorem 2, we can know that there exist a unique function $w \in W^{2,q}(\Omega)$ and a constant $c_7 > 0$ which depends only on n, p, Ω, and not on v such that

$$\begin{cases} L[\nabla v](w) = H(x,v,\nabla v) \quad \text{in } \Omega, \\ w - \phi \in W_0^{1,q}(\Omega), \\ \|w\|_{W^{2,q}(\Omega)} \le c_7(\|G\|_{W^{1,p}(\Omega\times\mathbb{R})} + \|\phi\|_{W^{2,q}(\Omega)}). \end{cases} \tag{27}$$

By the Sobolev inequality and (27), we obtain

$$\begin{aligned} \|w\|_{C^{1,1/2-n/2q}(\bar{\Omega})} &\le c_8\|w\|_{C^{1,1-n/q}(\bar{\Omega})} \\ &\le c_9\|w\|_{W^{2,q}(\Omega)} \\ &\le c_{10}(\|G\|_{W^{1,p}(\Omega\times\mathbb{R})} + \|\phi\|_{W^{2,q}(\Omega)}), \end{aligned} \tag{28}$$

where $c_8, c_9, c_{10} > 0$ depend only on n, p, and Ω. Suppose that

$$\|G\|_{W^{1,p}(\Omega\times\mathbb{R})} + \|\phi\|_{W^{2,q}(\Omega)} \le c_{10}^{-1} =: \delta_2(n,p,\Omega). \tag{29}$$

Let us define an operator $T : \mathscr{A} \to \mathscr{A}$ by $T(v) = w$ which satisfies (27). We show that $T(\mathscr{A})$ is precompact and T is a continuous mapping. For any sequence $\{v_m\}_{m\in\mathbb{N}} \subset \mathscr{A}$, we have $\sup_{m\in\mathbb{N}}\|T(v_m)\|_{C^{1,1-n/q}(\bar{\Omega})} \le c_8^{-1}$ by (28) and (29). There exists a subsequence $\{T(v_{m_k})\}_{k\in\mathbb{N}} \subset \{T(v_m)\}_{m\in\mathbb{N}}$ which converges to a function $w_\infty \in C^1(\bar{\Omega})$ in the sense of $C^1(\bar{\Omega})$ by the Ascoli-Arzelà theorem. We see that $w_\infty \in C^{1,1-n/q}(\bar{\Omega})$ because

$$\frac{|\nabla w_\infty(x) - \nabla w_\infty(y)|}{|x-y|^{1-n/q}} = \lim_{k\to\infty} \frac{|\nabla T(v_{m_k})(x) - \nabla T(v_{m_k})(y)|}{|x-y|^{1-n/q}} \le c_8^{-1}.$$

Let $\tilde{w}_k := T(v_{m_k}) - w_\infty$, and $\tilde{w}_k$ converges to 0 in the sense of $C^1(\bar{\Omega})$. Then we have

$$\begin{aligned}\frac{|\nabla \tilde{w}_k(x) - \nabla \tilde{w}_k(y)|}{|x-y|^{1/2-n/2q}} &\le \left(\frac{|\nabla \tilde{w}_k(x) - \nabla \tilde{w}_k(y)|}{|x-y|^{1-n/q}}\right)^{1/2} |\nabla \tilde{w}_k(x) - \nabla \tilde{w}_k(y)|^{1/2} \\ &\le 2c_8^{-1/2}(2\|\nabla \tilde{w}_k\|_{L^\infty(\Omega)})^{1/2}. \qquad (30)\end{aligned}$$

Hence, $\{T(v_{m_k})\}_{k\in\mathbb{N}}$ converges to a function w_∞ in the sense of $C^{1,1/2-n/2q}(\bar{\Omega})$, and the operator T is a compact mapping. In particular, the set $T(\mathscr{A})$ is precompact.

Suppose that $\{v_m\}_{m\in\mathbb{N}}$ converges to v in the sense of $C^{1,1/2-n/2q}(\bar{\Omega})$. By (28) and (29), $\sup_{m\in\mathbb{N}}\|T(v_m)\|_{W^{2,q}(\Omega)}$ is bounded. Hence, there exists a subsequence $\{T(v_{m_k})\}_{k\in\mathbb{N}} \subset \{T(v_m)\}_{m\in\mathbb{N}}$ which weakly converges to a function $w \in W^{2,q}(\Omega)$. We show $T(v) = w$, that is,

$$A_{ij}(\nabla v(x))D_{ij}w(x) = H(x, v, \nabla v).$$

For any $\psi \in C_0^\infty(\Omega)$, by the weak convergence and the Hölder inequality,

$$\begin{aligned}&\left|\int_\Omega \psi\{A_{ij}(\nabla v)D_{ij}w - A_{ij}(\nabla v_{m_k})D_{ij}(T(v_{m_k}))\}\right| \\ &\quad\le \left|\int_\Omega \psi A_{ij}(\nabla v)(D_{ij}w - D_{ij}(T(v_{m_k})))\right| \\ &\qquad + \left|\int_\Omega \psi D_{ij}(T(v_{m_k}))(A_{ij}(\nabla v) - A_{ij}(\nabla v_{m_k}))\right| \\ &\quad\le \left|\int_\Omega \psi A_{ij}(\nabla v)(D_{ij}w - D_{ij}(T(v_{m_k})))\right| \\ &\qquad + \|T(v_{m_k})\|_{W^{2,q}(\Omega)}\|\psi(A_{ij}(\nabla v) - A_{ij}(\nabla v_{m_k}))\|_{L^{q/(q-1)}(\Omega)} \\ &\quad\to 0 \qquad (k\to\infty). \qquad (31)\end{aligned}$$

By (24) and $\|v_{m_k}\|_{L^\infty(\Omega)}$, $\|v\|_{L^\infty(\Omega)} \le 1$, we compute

$$\begin{aligned}
&|H(x, v_{m_k}(x), \nabla v_{m_k}(x))| \\
&\quad \le |G(x, v_{m_k}(x)) - G(x, v(x))| + |G(x, v(x))| \\
&\quad \le \int_{-1}^{1} |D_t G(x, t)| dt + |G(x, v(x))|. \qquad (32)
\end{aligned}$$

$\int_{-1}^{1} |D_t G(\cdot, t)| dt + |G(\cdot, v(\cdot))|$ is an integrable function by Lemma 1, $\|v\|_{C^1(\bar{\Omega})} \le 1$, and Fubini's theorem. Since H is a continuous function with respect to t and z, using the dominated convergence theorem, we have

$$\int_\Omega \psi\{H(x, v(x), \nabla v(x)) - H(x, v_{m_k}(x), \nabla v_{m_k}(x))\} \to 0 \qquad (k \to \infty). \quad (33)$$

By (31) and (33),

$$\begin{aligned}
&\int_\Omega \psi\{A_{ij}(\nabla v) D_{ij} w - H(x, v(x), \nabla v(x))\} \\
&\quad = \lim_{k\to\infty} \int_\Omega \psi\{A_{ij}(\nabla v_{m_k}) D_{ij}(T(v_{m_k})) - H(x, v_{m_k}(x), \nabla v_{m_k}(x))\} \\
&\quad = 0. \qquad (34)
\end{aligned}$$

Using the fundamental lemma of the calculus of variations, we have

$$A_{ij}(x, \nabla v) D_{ij} w - H(x, v(x), \nabla v(x)) = 0 \qquad \text{for a.e. } x \in \Omega,$$

and $T(v) = w$. Hence, $\{T(v_m)\}_{m\in\mathbb{N}}$ weakly converges to $T(v)$ in $W^{2,q}(\Omega)$. By the compactness of T and the uniqueness of limit, we can show $\{T(v_m)\}_{m\in\mathbb{N}}$ converges to $T(v)$ in $C^{1,1/2-n/2q}(\overline{\Omega})$, and T is a continuous mapping. Using Theorem 4, we obtain a function $u \in W^{2,q}(\Omega)$ satisfying $u - \phi \in W_0^{1,q}(\Omega)$ and (25). □

Proof (Proof of Theorem 1). We should show that there exists a function $\tilde{u} \in W^{2,q}(\Omega)$ such that

$$A_{ij}(\nabla\tilde{u} + \nabla h) D_{ij}(\tilde{u} + h) = H(x, \tilde{u} + h, \nabla\tilde{u} + \nabla h), \qquad (35)$$

$$\tilde{u} - \phi \in W_0^{1,q}(\Omega), \qquad (36)$$

$$\|\tilde{u}\|_{W^{2,q}(\Omega)} < \varepsilon. \qquad (37)$$

Using the minimal surface equation (5) for h, we convert (35) as

$$\begin{aligned} A_{ij}(\nabla\tilde{u}+\nabla h)D_{ij}\tilde{u} &+ \frac{D_{ij}h}{(1+|\nabla\tilde{u}+\nabla h|^2)^{3/2}}((|\nabla\tilde{u}|^2+2\nabla\tilde{u}\cdot\nabla h)\delta_{ij} \\ &- D_i\tilde{u}D_j\tilde{u}-D_i\tilde{u}D_jh-D_j\tilde{u}D_ih) \\ &= H(x,\tilde{u}+h,\nabla\tilde{u}+\nabla h). \end{aligned} \tag{38}$$

Define

$$\mathscr{A} := \{v\in C^{1,1/2-n/2q}(\bar{\Omega});\ \|v\|_{C^{1,1/2-n/2q}(\bar{\Omega})}\le\varepsilon\}. \tag{39}$$

The set $\mathscr{A}$ is a closed convex set in Banach space $C^{1,1/2-n/2q}(\bar{\Omega})$. We consider the following differential equation,

$$\begin{aligned} A_{ij}(\nabla v+\nabla h)D_{ij}w &+ \frac{D_{ij}h}{(1+|\nabla v+\nabla h|^2)^{3/2}}((\nabla v\cdot\nabla w+2\nabla w\cdot\nabla h)\delta_{ij} \\ &- D_ivD_jw-D_iwD_jh-D_jwD_ih) \\ &= H(x,v+h,\nabla v+\nabla h). \end{aligned} \tag{40}$$

Define

$$\begin{aligned} B(\nabla v)\cdot\nabla w := \frac{D_{ij}h}{(1+|\nabla v+\nabla h|^2)^{3/2}}&((\nabla v\cdot\nabla w+2\nabla w\cdot\nabla h)\delta_{ij} \\ &- D_ivD_jw-D_iwD_jh-D_jwD_ih). \end{aligned}$$

Here, there exists a constant $c_{11}>0$ which depends only on n, p, Ω, ε, and $\|h\|_{W^{2,\infty}(\Omega)}$ such that

$$\|B_i(\nabla v)\|_{L^\infty(\Omega)}\le c_{11} \qquad \text{for all } i\in\{1,\dots,n\}, \tag{41}$$

where $B(\nabla v)=(B_1(\nabla v),\dots,B_n(\nabla v))\in L^\infty(\Omega;\mathbb{R}^n)$.

Using Theorem 2, we obtain a unique function $w\in W^{2,q}(\Omega)$ satisfying $w-\phi\in W_0^{1,q}(\Omega)$ and (40). By (41), Theorem 2, Lemma 1, and the Sobolev inequality, there exists a constant $c_{12}>0$ which depends only on n, p, Ω, ε, and $\|h\|_{W^{2,\infty}(\Omega)}$ such that

$$\|w\|_{C^{1,1/2-n/2q}(\bar{\Omega})}\le c_{12}(\|G\|_{W^{1,p}(\Omega\times\mathbb{R})}+\|\phi\|_{W^{2,q}(\Omega)}). \tag{42}$$

Suppose that we have

$$\|G\|_{W^{1,p}(\Omega\times\mathbb{R})}+\|\phi\|_{W^{2,q}(\Omega)}\le c_{12}^{-1}\varepsilon := \delta_1. \tag{43}$$

Let a operator $T:\mathscr{A}\to\mathscr{A}$ be defined by $T(v)=w$ which satisfies $w-\phi\in W_0^{1,q}(\Omega)$ and (40). The compactness of T can be proved by the argument of Theorem 5. In particular, the set $T(\mathscr{A})$ is precompact.

Suppose that $\{v_m\}_{m\in\mathbb{N}} \subset \mathscr{A}$ converges to v in the sense of $C^{1,1/2-n/2q}(\bar{\Omega})$. Then there exists a subsequence $\{T(v_{m_k})\}_{k\in\mathbb{N}} \subset \{T(v_m)\}_{m\in\mathbb{N}}$ which weakly converges to a function $w \in W^{2,q}(\Omega)$. For any $\psi \in C_0^\infty(\Omega)$,

$$\begin{aligned}
&\int_\Omega \psi\{B(\nabla v)\cdot\nabla w - B(\nabla v_{m_k})\cdot\nabla T(v_{m_k})\}\\
&\quad = \int_\Omega \psi B(\nabla v)\cdot(\nabla w - \nabla(T(v_{m_k})))\\
&\qquad + \int_\Omega \psi\nabla(T(v_{m_k}))\cdot(B(\nabla v) - B(\nabla v_{m_k}))\\
&\quad \to 0 \qquad (k\to\infty),
\end{aligned} \tag{44}$$

since B is a continuous function and $T(v_{m_k})$ converges weakly to w. By (44) and the argument of Theorem 5, we can show that T is a continuous mapping. Using Theorem 4, we obtain a function $\tilde{u} \in W^{2,q}(\Omega)$ satisfying (35) and (36). Moreover, $\tilde{u}$ satisfies (37) by (42) and (43). Define $u := \tilde{u} + h$. Then u satisfies $u - h - \phi \in W_0^{1,q}(\Omega)$, (8), and (9), and the proof is complete. □

Corollary 1. *Suppose* $f = (f_1, \ldots, f_{n+1}) \in W^{1,p}(\Omega\times\mathbb{R};\mathbb{R}^{n+1})$ *with* $\frac{n+1}{2} < p < n+1$ *and* $\phi \in W^{2,q}(\Omega)$ *with* $q = \frac{np}{n+1-p}$. *Let* $\varepsilon > 0$ *be arbitrary. Suppose* $h \in W^{2,\infty}(\Omega)$ *satisfies the minimal surface equation, that is,*

$$\operatorname{div}\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right) = 0. \tag{45}$$

Let $\delta_1 > 0$ *be the constant as in Theorem* 1. *If*

$$\sum_{i=1}^{n+1} \|f_i\|_{W^{1,p}(\Omega\times\mathbb{R})} + \|\phi\|_{W^{2,q}(\Omega)} \le \delta_1, \tag{46}$$

then there exists a function $u \in W^{2,q}(\Omega)$ *such that* $u - h - \phi \in W_0^{1,q}(\Omega)$ *and*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = v(\nabla u(x))\cdot f(x,u(x)) \qquad \text{in } \Omega, \tag{47}$$

$$\|u-h\|_{W^{2,q}(\Omega)} < \varepsilon. \tag{48}$$

Proof. Define

$$H(x,t,z) := v(z)\cdot f(x,t).$$

By $f \in W^{1,p}(\Omega \times \mathbb{R}; \mathbb{R}^{n+1})$, for a.e. $x \in \Omega$, $f(x,\cdot)$ is an absolutely continuous function. Hence $H(x,\cdot,\cdot)$ is a continuous function for almost every $x \in \Omega$. We have

$$|H(x,t,z)| \le \sum_{i=1}^{n+1} |f_i(x,t)| \qquad \text{for a.e. } x \in \Omega, \text{ any } (t,z) \in \mathbb{R} \times \mathbb{R}^n,$$

and $\sum_{i=1}^{n+1} |f_i(x,t)| \in W^{1,p}(\Omega \times \mathbb{R})$. By the Minkowski inequality,

$$\left\| \sum_{i=1}^{n+1} |f_i(x,t)| \right\|_{W^{1,p}(\Omega \times \mathbb{R})} \le \sum_{i=1}^{n+1} \|f_i\|_{W^{1,p}(\Omega \times \mathbb{R})}.$$

Define

$$G(x,t) := \sum_{i=1}^{n+1} |f_i(x,t)|.$$

Then H and G satisfy the assumption of Theorem 1, and this corollary follows. □

Remark 1. *The uniqueness of solutions follows immediately using* [4, Theorem 10.2]. *Under the assumptions of Theorem* 1, *if we additionally assume that H is non-decreasing in t for each $(x,z) \in \Omega \times \mathbb{R}^n$ and continuously differentiable with respect to the z variables in $\Omega \times \mathbb{R} \times \mathbb{R}^n$, then the solution is unique in $W^{2,q}(\Omega)$.*

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