

## A time-periodic oscillatory hexagonal solution in a 2-dimensional integro-differential reaction-diffusion system

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**ABSTRACT.** An oscillatory hexagonal solution in a two component reaction-diffusion system with a non-local term is studied. By applying the center manifold theory, we obtain a four-dimensional dynamical system that informs us about the bifurcation structure around the trivial solution. Our results suggest that the oscillatory hexagonal solution can bifurcate from a stationary hexagonal solution via the Hopf bifurcation. This provides a reasonable explanation for the existence of the oscillatory hexagon.

### 1. Introduction

**1.1. Preliminaries.** We study a pair of real-valued time-periodic solutions  $(u(t, x, y), v(t, x, y))$  in the integro-differential reaction-diffusion system:

$$\begin{cases} u_t = D_1 \Delta u + f(u, v) + \frac{s}{|\Omega|} \int_{\Omega} u \, dx dy, & t > 0, \\ v_t = D_2 \Delta v + g(u, v), & t > 0 \end{cases} \quad (1)$$

in a rectangular domain  $(x, y) \in \Omega := (0, L_1) \times (0, L_2) \subset \mathbb{R}^2$  under the Neumann boundary conditions:

$$\begin{cases} u_x(t, 0, y) = u_x(t, L_1, y) = 0, \\ u_y(t, x, 0) = u_y(t, x, L_2) = 0, \\ v_x(t, 0, y) = v_x(t, L_1, y) = 0, \\ v_y(t, x, 0) = v_y(t, x, L_2) = 0, \end{cases} \quad (2)$$

where  $L_1, L_2, D_1, D_2$ , and  $s$  are positive parameters,  $\Delta$  is the Laplacian, and  $f(u, v)$  and  $g(u, v)$  are sufficiently smooth functions. The system (1) is introduced as a mathematical model describing electrochemical experiments in [1, 2, 10]. When we put  $s = 0$ , the system (1) can be regarded as a so-called activator-inhibitor system with the following assumption:

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ASSUMPTION 1. *The nonlinear functions  $f(u, v)$  and  $g(u, v)$  satisfy the followings:*

$$(A1) \quad f(0, 0) = g(0, 0) = 0,$$

$$(A2) \quad f_u > 0, f_v < 0, g_u > 0, g_v < 0, f_u + g_v < 0, \delta := f_u g_v - f_v g_u > 0,$$

$$(A3) \quad \frac{f_v g_u}{g_v} + g_v < 0,$$

where  $f_u = \frac{\partial f}{\partial u}(0, 0)$  and so forth.

The assumptions (A1) and (A2) mean that the system (1) can exhibit the ‘‘Turing instability’’ at the trivial solution  $(u, v) = (0, 0)$  with  $s = 0$  ([6, 11]). That is, a spatially non-uniform stationary solution may appear from the spatially uniform solution. More precisely, the trivial solution is asymptotically stable in the sense of ordinary differential equations, however, it becomes unstable in the sense of partial differential equations by a suitable choice of  $D_1$  and  $D_2$ . The last assumption (A3) is required as a technical condition to guarantee that the center manifold to be constructed is attractive (see Section 2).

We also remark that the system (1) is the *shadow system* of the following three-component reaction-diffusion system:

$$\begin{cases} u_t = D_1 \Delta u + f(u, v) + sw, & (x, y) \in \Omega, t > 0, \\ v_t = D_2 \Delta v + g(u, v), & (x, y) \in \Omega, t > 0, \\ \tau w_t = D_3 \Delta w + u - w, & (x, y) \in \Omega, t > 0, \end{cases} \quad (3)$$

where  $D_3 > 0$  is the diffusion coefficient and the time constant  $\tau > 0$  is supposed to be very small. Under the limits  $D_3 \rightarrow \infty$  and  $\tau \rightarrow +0$ , (3) is formally reduced to (1). In fact, since the symmetry and multiplicity of the zero eigenvalues of (3) coincide with those of (1), the normal form (which is given by (9) in Section 3) derived from (1)–(2) is also derived from (3) based on the normal form theory. That is, the bifurcation structure of (3) under the Neumann boundary conditions around the trivial solution is similar to that of (1)–(2).

In the one dimensional space, the qualitative results of pattern dynamics, such as the wave bifurcation ([7]) for  $s < 0$  and chaotic dynamics ([4, 8, 9]) for  $s > 0$ , are obtained. In more detail, when  $s < 0$ , Ogawa [7] studied the system (1) under the periodic boundary conditions and found that a non-uniform time periodic oscillatory solution primarily bifurcates from the trivial solution by driving  $D_2$  in the case that the reaction terms  $f$  and  $g$  have the Hopf instability. Meanwhile, in the case  $s > 0$ , the bifurcation structures around the triply degenerate points for two spatially non-uniform modes and uniform one (0 : 1 : 2-mode interaction) were studied in [4, 8, 9]. In

particular, it was reported that a Hopf-zero instability is found at a non-trivial equilibrium that bifurcates from the trivial solution. Through this instability, a limit cycle, which bifurcates through the Hopf bifurcation from a non-trivial stationary solution, can undergo pitchfork bifurcation. This kind of bifurcation occurs if the Hopf bifurcation point and pitchfork bifurcation point overlap on the parameter space and is called a Hopf-pitchfork bifurcation. Moreover, it was also revealed that the Hopf-pitchfork bifurcation induced from the  $0:1:2$ -mode interaction leads to a torus, heteroclinic cycle and chaotic dynamics.

On the other hand, the results for the two-dimensional case of (1) have not yet been obtained. The numerical result shown in Figure 1 indicates the existence of the time-periodic oscillatory solution, and motivates us to study the dynamics and bifurcation structures induced by the basic wave numbers  $(0, 0)$ ,  $(1, \pm 1)$ , and  $(2, 0)$  in the two dimensional case. In this paper, we deal with the case that  $s > 0$ . For  $s < 0$ , since the wave bifurcation may be induced by a suitable choice of parameters, it is necessary to analyze a normal form other than (9) (see Section 3) to obtain the bifurcation structures, hence, we leave it open here.

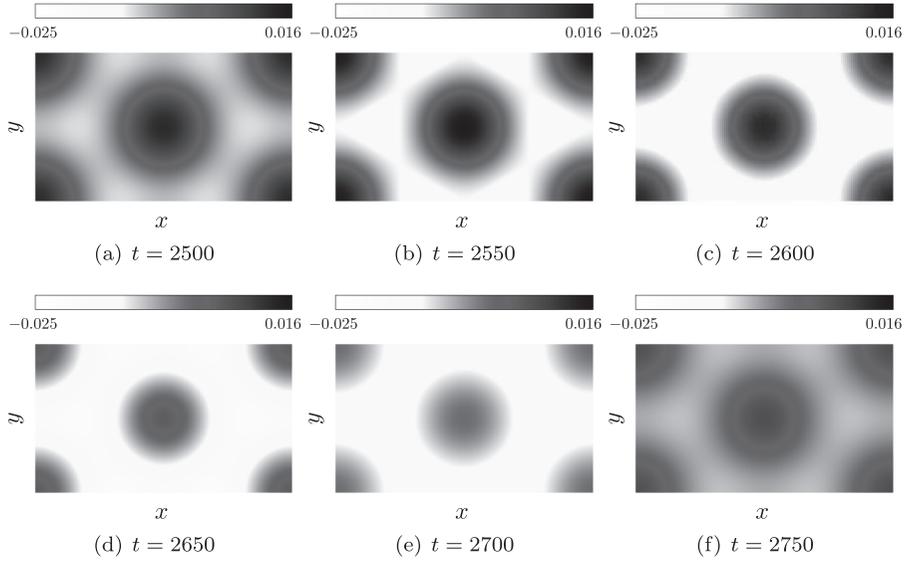
**1.2. Numerical examples.** If we set

$$\begin{aligned} f(u, v) &= u - 10v + u^2 - u^3, & g(u, v) &= 2u - 5v + u^2, \\ D_1 &= \frac{2\sqrt{3} - 3}{4}, & L_1 &= \pi, & L_2 &= \frac{L_1}{\sqrt{3}}, \end{aligned} \quad (4)$$

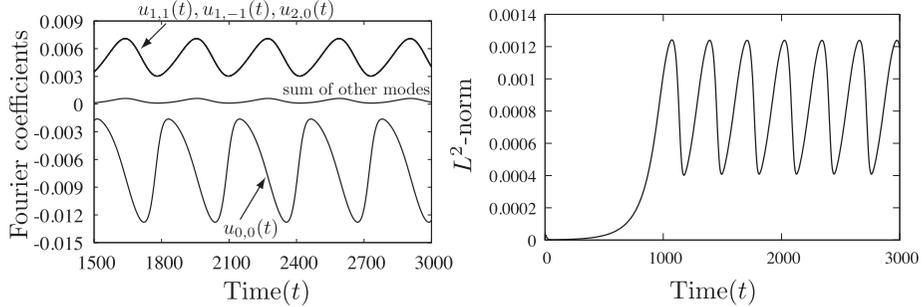
then the linearized operator for the trivial solution has multiple zero eigenvalues at

$$(s^*, D_2^*) = \left( 3, \frac{5(3 + 2\sqrt{3})}{4} \right),$$

which is called the multiply-degenerate point (see Definition 2 in Section 2). By taking the parameters near the multiply-degenerate point, such as  $(s, D_2) = (2.985, 8.192)$ , we can numerically find the time-periodic oscillatory hexagonal solution (see Figure 1). We also set the number of grid points and the time-mesh size as  $64 \times 64$  and  $1.0 \times 10^{-5}$ , respectively. Figure 1 shows the time evolution of  $u(t, x, y)$  at  $t \in [2500, 2750]$ . Figure 2 shows time evolutions of the Fourier coefficients and the  $L^2$ -norm of  $u(t, x, y)$  at  $t \in [0, 3000]$ . The hexagon (or hexagonal solution) we say in this paper is the solution whose level set forms regular hexagons. More precisely, the leading terms of the hexagonal solution are written by

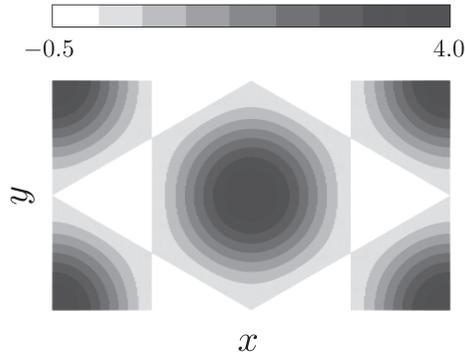


**Fig. 1.** Time evolution of the time-periodic oscillatory hexagonal solution in the case of (4). These figures show  $\bar{u}(t, x, y)$  at  $t \in [2500, 2750]$  for every 50 time steps.



**Fig. 2.** (Left): Time evolutions of the Fourier coefficients of the numerical solution  $u(t, x, y)$  of (1) and (2). Fourier  $(0, 0)$ ,  $(1, 1)$ ,  $(1, -1)$ ,  $(2, 0)$  modes are shown. The “sum of other modes” is  $\sum_{|m|, |n| \leq 64} u_{m, n}(t) - \sum' u_{m', n'}(t)$ , where  $\sum'$  stands for the summation of the critical modes. (Right):  $L^2$ -norm of  $u(t, x, y)$  at  $t \in [0, 3000]$ .

$$\begin{aligned}
 u(t, x, y) &= u_{0,0}(t) + u_{1,1}(t)\Phi_1(x; k)\Phi_1(y; l) \\
 &\quad + u_{1,-1}(t)\Phi_1(x; k)\Phi_{-1}(y; l) + u_{2,0}(t)\Phi_2(x; k), \\
 v(t, x, y) &= v_{0,0}(t) + v_{1,1}(t)\Phi_1(x; k)\Phi_1(y; l) \\
 &\quad + v_{1,-1}(t)\Phi_1(x; k)\Phi_{-1}(y; l) + v_{2,0}(t)\Phi_2(x; k),
 \end{aligned} \tag{5}$$



**Fig. 3.** A hexagonal pattern in the case where  $u_{0,0} = u_{1,1} = u_{1,-1} = u_{2,0} = 1$  in (5).

where  $u_{i,j} \in \mathbb{R}$ ,  $v_{i,j} \in \mathbb{R}$ ,  $\Phi_m(x; k) = \cos(mkx)$ ,  $\Phi_n(y; l) = \cos(nly)$ ,  $k = \pi/L_1$ , and  $l = \pi/L_2$  (see Figure 3).

The numerical result in Figure 1 shows the time-periodic oscillatory hexagonal solution of (1) expanded on  $\tilde{\Omega} := [0, 2L_1] \times [0, 2L_2]$  to clearly visualize the hexagonal patterns. More precisely, we define the new function  $\tilde{u}$  on  $\tilde{\Omega}$  using the solution  $u^*$  of (1) as follows:

$$\tilde{u}(t, x, y) = \begin{cases} u^*(t, x, y), & (x, y) \in \Omega, \\ u^*(t, 2L_1 - x, y), & (x, y) \in \Omega_1, \\ u^*(t, x, 2L_2 - y), & (x, y) \in \Omega_2, \\ u^*(t, 2L_1 - x, 2L_2 - y), & (x, y) \in \Omega_3, \end{cases} \quad (6)$$

where  $\Omega_1 := [L_1, 2L_1] \times [0, L_2]$ ,  $\Omega_2 := [0, L_1] \times [L_2, 2L_2]$ , and  $\Omega_3 := [L_1, 2L_1] \times [L_2, 2L_2]$ . We define the new function  $\tilde{v}$  on  $\tilde{\Omega}$ , similarly. Then the pair of functions  $(\tilde{u}(t, x, y), \tilde{v}(t, x, y))$  satisfies the system (1) via replacing  $\Omega$  by  $\tilde{\Omega}$  and the Neumann boundary conditions on  $\tilde{\Omega}$ . Conversely, if the pair of functions  $(\tilde{u}(t, x, y), \tilde{v}(t, x, y))$  is a solution of it, then we obtain the solution of (1) by restricting  $(\tilde{u}, \tilde{v})$  to  $\Omega$ .

Figure 2 supports that the leading terms of the numerical solution are given by (5) and that the amplitude of the other Fourier modes is sufficiently small. From the point of view of the local bifurcation theory, we expect that such solution bifurcates from a stationary hexagonal solution through the Hopf bifurcation, that is, the oscillatory hexagonal solution bifurcates as a secondary bifurcation from the trivial solution. Moreover, we can see that the Fourier  $(0, 0)$ ,  $(1, 1)$ ,  $(1, -1)$ , and  $(2, 0)$  modes become critical as shown in Figure 2, and therefore, it is necessary to investigate a multiply-degenerate point at which these four modes interact.

In general, a tedious amount of calculations are necessary to obtain the coefficients of the normal form explicitly. However, herein, we will focus on

several types of symmetry intrinsic to the original PDE system (1) and obtain the normal form without them.

This paper is organized as follows. In the next section, we study the instability at the trivial solution in a Fourier space to seek the primary bifurcation. We consider the case that the linearized operator around the trivial solution has the zero eigenvalues. To seek a multiple bifurcation point on which the linearized operator has multiple zero eigenvalues, we introduce “neutral stability surfaces”. Section 3 is devoted to obtaining a reduced system on the center manifold. This reduced system informs us about the bifurcation structure and dynamics around the trivial solution. Section 4 describes the necessary conditions for the Hopf bifurcation, which is the main result in this paper. Moreover, this result provides a good explanation for the existence of the time periodic oscillatory hexagonal solution as shown in Figures 1 and 2. Some remarks and future works are mentioned in the last section.

## 2. Linearized stability surfaces

We define the phase space  $\mathcal{X}$  of the dynamical system (1)–(2) as follows:

$$\mathcal{X} := \{(u, v) \in H^2(\Omega) \times H^2(\Omega); u \text{ and } v \text{ satisfy (2)}\}.$$

Substituting the Fourier expansions:

$$u(t, x, y) = \sum_{m, n \in \mathbb{Z}} u_{m, n}(t) \Phi_m(x; k) \Phi_n(y; l),$$

$$v(t, x, y) = \sum_{m, n \in \mathbb{Z}} v_{m, n}(t) \Phi_m(x; k) \Phi_n(y; l)$$

into (1) and using the orthogonality of trigonometric functions, we obtain the following infinite dimensional dynamical system:

$$\begin{pmatrix} \dot{u}_{m, n} \\ \dot{v}_{m, n} \end{pmatrix} = M_{m, n} \begin{pmatrix} u_{m, n} \\ v_{m, n} \end{pmatrix} + \begin{pmatrix} F_{m, n} \\ G_{m, n} \end{pmatrix}, \quad (7)$$

where  $m$  and  $n$  are integers,

$$M_{0, 0} = \begin{pmatrix} f_u + s & f_v \\ g_u & g_v \end{pmatrix},$$

$$M_{m, n} = \begin{pmatrix} f_u - D_1 \omega_{m, n}^2(k, l) & f_v \\ g_u & g_v - D_2 \omega_{m, n}^2(k, l) \end{pmatrix},$$

$$\omega_{m, n}^2(k, l) := m^2 k^2 + n^2 l^2,$$

and  $F_{m,n}$  and  $G_{m,n}$  are higher order terms with respect to  $u_{m,n}$  and  $v_{m,n}$ . We define the phase space of (7) by

$$\mathcal{X}_F := \left\{ \{(u_{m,n}, v_{m,n})\}_{(m,n) \in \mathbb{Z}^2}; \|\{(u_{m,n}, v_{m,n})\}\|_{\mathcal{X}_F}^2 = \sum_{m,n \in \mathbb{Z}} (1 + m^2 + n^2)^2 |(u_{m,n}, v_{m,n})|^2 < \infty \right\}.$$

Under this setting, the linearized operator of (1)–(2) is a generator of an analytic semigroup. To study the bifurcation structure around the trivial solution  $(u, v) \equiv (0, 0)$ , it is convenient to introduce the neutral stability surfaces:

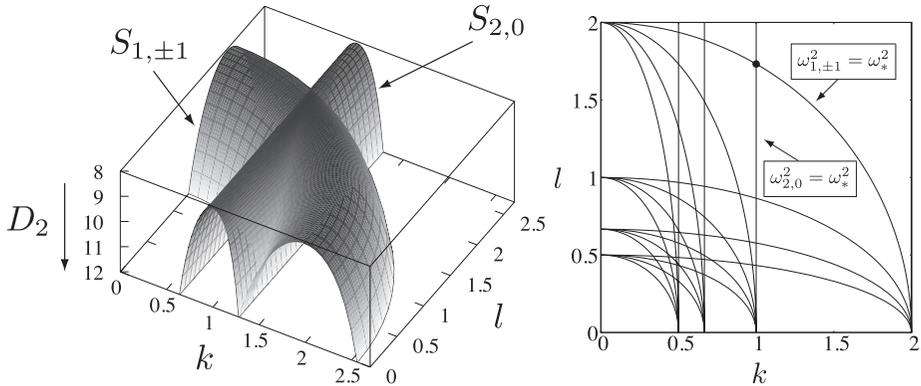
**DEFINITION 1.** *We call the set of parameters  $(D_2, k, l)$ , which satisfy  $\text{Det } M_{m,n} = 0$  as the neutral stability surfaces.*

More precisely, the neutral stability surfaces  $S_{m,n}$  are given by

$$S_{m,n} = \left\{ (D_2, k, l) \in \mathbb{R}^3; D_2(k, l) := \frac{g_v D_1 \omega_{m,n}^2 - \delta}{\omega_{m,n}^2 (D_1 \omega_{m,n}^2 - f_u)} \right\}. \tag{8}$$

In addition, we find that the minimal values of  $D_2(k, l)$  are the same for all  $(m, n) \setminus \{(0, 0)\}$  by simple computation. The minimal value is given by

$$D_2^* = \frac{-g_v^2 D_1 \sqrt{\delta^2 - f_u g_v \delta}}{2\delta(\delta - f_u g_v) - (2\delta - f_u g_v) \sqrt{\delta^2 - f_u g_v \delta}}.$$



**Fig. 4.** (Left): Neutral stability surfaces  $S_{2,0}$  and  $S_{1,\pm 1}$ . (Right): Contours  $\omega_{m,n}(k, l)^2 = \omega_*^2 = 4$  for  $m = 1, 2, \dots, 4, n = 0, 1, \dots, 4$ . The neutral stability surfaces have minimal value on these lines, and the intersection of  $\omega_{1,\pm 1}^2 = \omega_{2,0}^2$  is  $(k, l) = (1, \sqrt{3})$ , which is displayed with “•”. Both figures are written in the case of (4).

at  $\omega_{m,n}^2(k,l) = \omega_*^2 := (\delta - \sqrt{\delta^2 - f_u g_v \delta}) / (g_v D_1)$ . Here it should be noted that if we take  $(k,l) = (1, \sqrt{3})$  and  $L_1 = \sqrt{3}L_2 = \pi$ , then it follows that  $\omega_*^2 = 4$  and  $D_1 = (2\sqrt{3} - 3)/4$ . These values are used for the numerical experiments in Section 1.2. We define the multiple degenerate points by the intersection of the surfaces:

**DEFINITION 2.** *For a given pair of natural numbers  $(m,n)$ , the triplet of parameters  $(s^*, D_2^*, \omega_*^2)$  at which the linearized matrices  $M_{0,0}$  and  $M_{m,n}$  simultaneously have a simple zero eigenvalue, is called multiply degenerate point.*

Throughout this paper, we restrict our attention to the case that  $L_1 = \sqrt{3}L_2$ . We then obtain the following.

**PROPOSITION 1.** *Assume  $L_1 = \sqrt{3}L_2$ . Then, there exists a multiply-degenerate point such that the linearized matrices  $M_{0,0}$ ,  $M_{1,1}$ ,  $M_{1,-1}$ , and  $M_{2,0}$  simultaneously have a simple zero eigenvalue.*

### 3. Normal form on the center manifolds

We note that the system (1) has the symmetry properties.

**PROPOSITION 2.** *The followings hold:*

- (i) *The system (1) itself is invariant under the mappings  $x \mapsto x + \eta_1$  and  $y \mapsto y + \eta_2$  ( $\forall \eta_1, \forall \eta_2$ ).*
- (ii) *The system (1)–(2) is invariant under the mappings  $x \mapsto -x$  and  $y \mapsto -y$ .*

Then, the system (1) possesses symmetries represented by  $\tau_\eta$  and  $\mathcal{S}$ , defined by

$$\begin{aligned} (\tau_\eta U)(t, X) &= U(t, X + \eta), & \forall \eta \in \mathbb{R}^2 \\ (\mathcal{S}U)(t, X) &= U(t, -X), \end{aligned}$$

where  $U = (u, v)$  and  $X = (x, y)$ . In addition, in the case of  $L_1 : L_2 = \sqrt{3} : 1$ , the regular hexagonal pattern (including the oscillatory one) is invariant under the rotation

$$(\mathcal{R}_{\pi/3}U)(t, X) = (U(t, R_{\pi/3}X)),$$

where  $R_{\pi/3}$  is the rotation through angle  $\pi/3$  in  $xy$ -plain. Moreover, note that the unknown functions  $u$  and  $v$  are real functions. In what follows, we compute the normal form by using these symmetries.

Consider the complex Fourier expansion of  $U$  associated with (1):

$$U = \sum_{m,n \in \mathbb{Z}} U_{m,n}(t) e^{i(mkx + nly)}, \quad U_{m,n}(t) \in \mathbb{C}^2$$

with  $U_{m,n} = \bar{U}_{-m,-n}$ . Let  $\tilde{u}_{m,n}(t) \in \mathbb{C}$  be the Fourier coefficients that correspond to the center eigenspaces. We then consider the symmetries that are inherited by the normal form as follows:

- $\mathcal{S}(\tilde{u}_{0,0}, \tilde{u}_{1,1}, \tilde{u}_{1,-1}, \tilde{u}_{2,0}) = (\tilde{u}_{0,0}, \tilde{u}_{-1,-1}, \tilde{u}_{-1,1}, \tilde{u}_{-2,0})$ .
- $\mathcal{R}_{\pi/3}(\tilde{u}_{0,0}, \tilde{u}_{1,1}, \tilde{u}_{1,-1}, \tilde{u}_{2,0}) = (\tilde{u}_{0,0}, \tilde{u}_{2,0}, \tilde{u}_{1,1}, \tilde{u}_{1,-1})$ .
- Let  $\Theta_{m,n} = (mk, nl)$ . Then, for a given  $\eta \in \mathbb{R}^2$ , the normal form inherits the following symmetry:

$$\tau_\eta(\tilde{u}_{0,0}, \tilde{u}_{1,1}, \tilde{u}_{1,-1}, \tilde{u}_{2,0}) = (\tilde{u}_{0,0}, e^{i\Theta_{1,1}\eta} \tilde{u}_{1,1}, e^{i\Theta_{1,-1}\eta} \tilde{u}_{1,-1}, e^{i\Theta_{2,0}\eta} \tilde{u}_{2,0}).$$

Now we can derive the normal form on the center manifold formally by considering the symmetries  $\tau_\eta$ ,  $\mathcal{S}$  and  $\mathcal{R}_{\pi/3}$  with taking the boundary conditions (2) into account. The boundary conditions (2) require that  $U_{m,n} \in \mathbb{R}^2$  holds (or equivalently,  $U_{m,n} = \bar{U}_{m,n}$ ). It should be noted that the restriction on the real space corresponds to the restriction of  $U$  to the pair of functions  $U|_{\bar{\Omega}}$  that satisfies the boundary conditions (2) on  $\partial\Omega$ .

Then, by decomposing (7) into the stable and center eigenspaces with

$$\begin{pmatrix} \alpha_{i,j} \\ \beta_{i,j} \end{pmatrix} = T_{i,j}^{-1} \begin{pmatrix} u_{i,j} \\ v_{i,j} \end{pmatrix} \quad ((i,j) = (0,0), (1,\pm 1), (2,0)),$$

$$T_{0,0} = \begin{pmatrix} -g_v & f_v \\ g_u & g_v \end{pmatrix}, \quad T_{1,\pm 1} = T_{2,0} = \begin{pmatrix} -f_v & f_u - D_1 \omega_*^2 \\ f_u - D_1 \omega_*^2 & g_u \end{pmatrix},$$

we formally obtain the normal form of (7) at the quadruply degenerate point given in Proposition 1:

$$\left\{ \begin{array}{l} \dot{\alpha}_{0,0} = \mu_0 \alpha_{0,0} + A_1 \alpha_{0,0}^2 + A_2 (\alpha_{1,1}^2 + \alpha_{1,-1}^2 + \alpha_{2,0}^2) \\ \quad + (a_1 \alpha_{0,0}^2 + a_2 \alpha_{1,1}^2 + a_2 \alpha_{1,-1}^2 + a_2 \alpha_{2,0}^2) \alpha_{0,0} + a_3 \alpha_{1,1} \alpha_{1,-1} \alpha_{2,0} + \mathcal{O}_4, \\ \dot{\alpha}_{1,1} = \mu \alpha_{1,1} + B_1 \alpha_{0,0} \alpha_{1,1} + B_2 \alpha_{1,-1} \alpha_{2,0} \\ \quad + (b_1 \alpha_{0,0}^2 + b_2 \alpha_{1,1}^2 + b_3 \alpha_{1,-1}^2 + b_3 \alpha_{2,0}^2) \alpha_{1,1} + b_4 \alpha_{0,0} \alpha_{1,-1} \alpha_{2,0} + \mathcal{O}_4, \\ \dot{\alpha}_{1,-1} = \mu \alpha_{1,-1} + B_1 \alpha_{0,0} \alpha_{1,-1} + B_2 \alpha_{1,1} \alpha_{2,0} \\ \quad + (b_1 \alpha_{0,0}^2 + b_3 \alpha_{1,1}^2 + b_2 \alpha_{1,-1}^2 + b_3 \alpha_{2,0}^2) \alpha_{1,-1} + b_4 \alpha_{0,0} \alpha_{1,1} \alpha_{2,0} + \mathcal{O}_4, \\ \dot{\alpha}_{2,0} = \mu \alpha_{2,0} + B_1 \alpha_{0,0} \alpha_{2,0} + B_2 \alpha_{1,1} \alpha_{1,-1} \\ \quad + (b_1 \alpha_{0,0}^2 + b_3 \alpha_{1,1}^2 + b_3 \alpha_{1,-1}^2 + b_2 \alpha_{2,0}^2) \alpha_{2,0} + b_4 \alpha_{0,0} \alpha_{1,1} \alpha_{1,-1} + \mathcal{O}_4, \end{array} \right. \quad (9)$$

where we set  $\mu_0 := \mu_{0,0}$  and  $\mu := \mu_{1,\pm 1} = \mu_{2,0}$  with

$$\mu_{i,j} = \frac{\text{Tr } M_{i,j} + \sqrt{(\text{Tr } M_{i,j})^2 - 4 \text{Det } M_{i,j}}}{2}.$$

Furthermore,  $a_j, b_j, A_j, B_j \in \mathbb{R}$  are constants and  $\mathcal{O}_4$  denote  $\mathcal{O}(|(\alpha_{0,0}, \alpha_{1,1}, \alpha_{1,-1}, \alpha_{2,0})|^4)$ .

The dynamical system (9) is invariant under the transformation

$$(\alpha_{0,0}, \alpha_{1,1}, \alpha_{1,-1}, \alpha_{2,0}) \mapsto (\alpha_{0,0}, -\alpha_{1,1}, -\alpha_{1,-1}, \alpha_{2,0})$$

and exchange of  $\alpha_{1,1}$ ,  $\alpha_{1,-1}$ , and  $\alpha_{2,0}$ . These properties are derived from the symmetries of  $\tau_\eta$ ,  $\mathcal{S}$  and  $\mathcal{R}_{\pi/3}$ . In addition, if we take the parameters as  $(s, D_2, k, l) = (s^*, D_2^*, 1, \sqrt{3})$ , then the center manifold associated with the Fourier  $(0, 0)$ ,  $(1, 1)$ ,  $(1, -1)$ , and  $(2, 0)$  modes is attractive.

**REMARK 1.** *As mentioned above, the form of normal form is determined by the symmetry properties of the original partial differential equations. However, we cannot determine the values of coefficients in the normal form in general. To determine the values of coefficients, we have to approximate the center manifold according to the nonlinear terms  $f(u, v)$  and  $g(u, v)$ . Then, for a given set of parameters, we can compute the value of coefficients as in [4, 8].*

#### 4. Hopf bifurcation

It is convenient to restrict the dynamical system (9) on the phase space  $\mathcal{I} = \{(\alpha_{0,0}, \alpha_{1,1}, \alpha_{1,-1}, \alpha_{2,0}); \alpha_{1,1} = \alpha_{1,-1} = \alpha_{2,0} = \beta\}$  that is invariant under the flow of (9). Indeed, since the dynamical system (9) has the invariance under the symmetries  $\tau_\eta$ ,  $\mathcal{S}$  and  $\mathcal{R}_{\pi/3}$ , and the uniqueness of the solution, the trajectory does not leave this space if it starts from a point on  $\mathcal{I}$ . Therefore, the invariant set (equilibrium or limit cycle) on restricted phase space  $\mathcal{I}$  is also the invariant set of the dynamical system (9). In what follows, we consider the restricted planar system up to the cubic terms:

$$\begin{cases} \dot{\alpha} = \mu_0\alpha + A_1\alpha^2 + 3A_2\beta^2 + (a_1\alpha^2 + 3a_2\beta^2)\alpha + a_3\beta^3, \\ \dot{\beta} = \mu\beta + B_1\alpha\beta + B_2\beta^2 + \{b_1\alpha^2 + (b_2 + 2b_3)\beta^2\}\beta + b_4\alpha\beta^2, \end{cases} \quad (10)$$

where we put  $\alpha_{0,0} = \alpha$ . The equilibrium  $(\alpha^*, \beta^*)$  satisfying  $\beta^* \neq 0$  of (10) corresponds to the stationary regular hexagonal pattern. However, since we are interested in oscillatory (non-stationary) one, we seek an equilibrium that has a Hopf instability point. To do this, we consider an equilibrium which has a specific form via introducing new parameter  $\rho \in \mathbb{R} \setminus \{0\}$  as  $(\alpha, \beta) = (\alpha, \rho\alpha)$ . The linearized matrix is given by

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix},$$

where

$$\begin{aligned} m_{11} &= A_1\alpha + 2a_1\alpha^2 - 3A_2\beta^2/\alpha - a_3\beta^3/\alpha, \\ m_{12} &= 3(2A_2 + 2a_2\alpha + a_3\beta)\beta, \\ m_{21} &= (B_1 + 2b_1\alpha + b_4\beta)\beta, \\ m_{22} &= \{B_2 + 2(b_2 + 2b_3)\beta + b_4\alpha\}\beta. \end{aligned}$$

If  $\text{Tr } M = 0$  and  $\text{Det } M > 0$  hold, then the system (10) has a Hopf instability, that is, the following result is obtained:

**THEOREM 1.** *Assume that  $B_2^2 + 6A_2B_1 < 0$  holds. Set  $\rho \in \mathbb{R} \setminus \{0\}$  so that the following holds:*

$$A_1 + 2a_1\alpha + (B_2 + b_4\alpha)\rho + \{2(b_2 + 2b_3)\alpha - 3A_2\}\rho^2 - a_3\alpha\rho^3 = 0.$$

Then the linearized matrix  $M$  around the equilibrium  $(\alpha^*, \beta^*)$  has a pair of purely imaginary eigenvalues at  $(\mu_0, \mu) = (\mu_0^*, \mu^*)$ , where

$$\begin{aligned} \alpha^* &= \frac{3A_2\rho^2 - B_2\rho - A_1}{2a_1 + b_4\rho + 2(b_2 + 2b_3)\rho^2 - a_3\rho^3}, & \beta^* &= \rho\alpha^*, \\ \mu_0^* &= -[\rho^2\{3a_2 + 2(b_2 + 2b_3)\} + \rho b_4 + 3a_1](\alpha^*)^2 - (2A_1 + \rho B_2)\alpha^*, \\ \mu^* &= -\{(b_1 + \rho b_4 + \rho^2 b_2 + 2\rho^2 b_3)\alpha^* + B_1 + \rho B_2\}\alpha^*. \end{aligned}$$

**REMARK 2.** *We can obtain a simple formula of the set of equilibrium and parameters for the Hopf bifurcation by introducing  $\rho$  as above. The relationship of the parameters  $\rho$  and  $(s, D_2)$  is as follows. From the straightforward computations, the equilibrium of (10) has the form*

$$\begin{cases} \alpha = \alpha^*(\mu_0(s), \mu(D_2)), \\ \beta = \beta^*(\mu_0(s), \mu(D_2)). \end{cases}$$

Then,  $\rho$  has the formula

$$\rho = \frac{\beta^*(\mu_0(s), \mu(D_2))}{\alpha^*(\mu_0(s), \mu(D_2))}$$

and the Hopf bifurcation points are lie on the set

$$\{(\mu_0(s), \mu(D_2)) \in \mathbb{R}^2 \mid \text{Tr } M(\mu_0(s), \mu(D_2)) = 0\}.$$

Therefore, we take  $\rho \in \mathbb{R} \setminus \{0\}$  on the set

$$\left\{ \rho \mid \rho = \frac{\beta^*(\mu_0(s), \mu(D_2))}{\alpha^*(\mu_0(s), \mu(D_2))}, \text{Tr } M(\mu_0(s), \mu(D_2)) = 0 \right\}$$

to consider the Hopf bifurcation.

REMARK 3. Note that if the free parameter  $\rho$  tends to the solution of the following equation:

$$3A_2\rho^2 - B_2\rho - A_1 = 0,$$

then the bifurcation parameters  $\mu_0^*$ ,  $\mu^*$  and equilibrium converge to the origin. That is, the equilibrium  $(\alpha^*, \beta^*)$  and the dynamics around it can be constrained on the center manifold with a suitable choice of parameters.

REMARK 4. At the point  $(\mu_0, \mu) = (\mu_0^*, \mu^*)$ , the linearized eigenvalues of (9) coincide with those of (10). Notice that the remaining ones are given by

$$A := 2\beta^*((b_2\beta^* - b_3)\beta^* - b_4\alpha^* - B_2).$$

We then derive the normal form of the Hopf bifurcation. To translate the equilibrium to the origin, we set  $\alpha - \alpha^* = X$  and  $\beta - \beta^* = Y$ . Then, the system (10) is given by

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & -m_{11} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} F(X, Y) \\ G(X, Y) \end{pmatrix},$$

where

$$\begin{aligned} F(X, Y) &= (A_1 + 3a_1\alpha^*)X^2 + 3(A_2 + a_2\alpha^* + a_3\beta^*)Y^2 \\ &\quad + 6a_2\beta^*XY + a_1X^3 + a_3Y^3 + 3a_2XY^2, \\ G(X, Y) &= b_1\beta^*X^2 + \{B_2 + 3\beta^*(b_2 + 2b_3) + b_4\alpha^*\}Y^2 \\ &\quad + (B_1 + 2b_1\alpha^* + 2b_4\beta^*)XY + (b_2 + 2b_3)Y^3 + b_1X^2Y + b_4XY^2. \end{aligned}$$

The eigenvalues of  $M$  are  $\pm i\omega$ , where  $\omega = \sqrt{-m_{11}^2 - m_{12}m_{21}}$ . At the Hopf instability point, the dynamical system is transformed into the normal form of the Hopf bifurcation. The following is one of the main results of this paper:

THEOREM 2. If the inequality  $B_2^2 + 6A_2B_1 < 0$  and  $d(\text{Tr } M)/d\rho \neq 0$  are satisfied at the Hopf instability point, then there exists a constant  $C_h \in \mathbb{C}$  such that the dynamical system (10) around the equilibrium  $(\alpha^*, \beta^*)$  is transformed into the following complex ordinary differential equation:

$$\dot{z} = \lambda z + C_h |z|^2 z + \mathcal{O}(|z|^4), \quad z(t) \in \mathbb{C},$$

where  $\lambda = i\omega$ .

The algorithm for the calculation of  $C_h$  and the normal form for the Hopf bifurcation are given in [5]. It is well known that the sign of  $\text{Re}\{C_h\}$  determines the stability of the periodic orbit.

**PROPOSITION 3.** *If  $\text{Re}\{C_h\}$  and  $A$  are negative then the system (1) has a locally asymptotically stable small-amplitude time-periodic solution that bifurcates from the stationary hexagonal solution corresponding to the equilibrium  $(\alpha_{0,0}, \alpha_{1,1}, \alpha_{1,-1}, \alpha_{2,0}) = (\alpha^*, \beta^*, \beta^*, \beta^*)$  of (9) through the Hopf bifurcation.*

Indeed,  $C_h$  depends on the parameters and constants in (1), however, if  $\text{Re}\{C_h\} \neq 0$ , then the sign of it is determined for the parameters in a neighborhood of the Hopf bifurcation point. More precisely, there exists a positive constant  $\varepsilon$  such that if  $(s, D_2)$  satisfies  $\|(\mu_0(s), \mu(D_2)) - (\mu_0^*, \mu^*)\| < \varepsilon$ , then  $\text{sign}\{\text{Re}\{C_h(\mu_0(s), \mu(D_2))\}\} = \text{sign}\{\text{Re}\{C_h(\mu_0^*, \mu^*)\}\}$ . In addition, for the fixed  $D_1, L_1$ , and  $L_2 = L_1/\sqrt{3}$ , we can drive  $s$  and  $D_2$  by controlling the parameter  $\rho$  so that  $\mu_0(s)$  and  $\mu(D_2)$  are in a neighborhood of the Hopf bifurcation point  $(\mu_0^*, \mu^*)$ . Moreover,  $\lambda \in i\mathbb{R}$  holds at the Hopf bifurcation point  $(\mu_0(s), \mu(D_2)) = (\mu_0^*, \mu^*)$ . Therefore, if the assumptions of Theorem 2 hold, then the Hopf bifurcation occurs with moving parameters  $(s, D_2)$  near the Hopf bifurcation point.

### 5. Concluding remarks

In this paper, we studied the Hopf bifurcation from the regular hexagonal solution in the integro-differential reaction-diffusion system (1)–(2). By setting  $D_2, L_1, L_2$ , and  $s$  appropriately, the linearized operator around the trivial solution has quadruply zero eigenvalues. In particular, we formally obtained the four-dimensional dynamical system (9) on the center manifold by focusing our attention on the Fourier  $(0, 0) : (1, 1) : (1, -1) : (2, 0)$  mode interaction. Furthermore, the necessary conditions for the Hopf bifurcation around the stationary hexagonal solution were obtained. We note that if  $(0, 0)$ -mode is not a critical mode, that is, the parameter  $s$  is set far from the bifurcation point  $s = s^*$ , then the dynamical system on the center manifold is given by

$$\dot{\beta} = \mu\beta + a\beta^2 + b\beta^3 + \mathcal{O}(|\beta|^4), \tag{11}$$

where  $a$  and  $b$  are constants. Then, the stationary hexagonal solution cannot undergo a Hopf bifurcation in this situation, therefore, this fact emphasizes the importance of the bifurcation parameter  $s$ .

To give validity to the numerical results (Figures 1 and 2), which imply  $\text{Re}\{C_h\} < 0$ , we will determine the explicit forms of  $C_h$  as well as all coefficients of reduced system (9) by more detailed computations in our future works. In addition, the dynamics around the quadruply-degenerate point whose modes are  $(0, 0) : (i, j) : (i, -j) : (h, 0)$ , where  $h/i \notin \mathbb{Z}$  should be studied, and will be addressed in future works as well.

As we mentioned in Section 1, the reduced system (9) is also derived from the three-component system (3) but cannot possess the Hopf instability point in (9) for  $s = 0$  (i.e., the activator-inhibitor case). In contrast, the results given in the present paper suggest that the three-component system (3) can possess oscillatory hexagonal solutions in the case that  $s > 0$ ,  $D_3 \gg 1$ , and  $0 < \tau \ll 1$ .

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