Hiroshima Math. J.
50 (2020), 169–184

Pointwise multipliers on weak Orlicz spaces

Ryota Kawasumi and Eiichi Nakai

(Received May 20, 2018)
(Revised March 7, 2020)

Abstract. It is well known that the set of all functions g such that "$f \in L^{p_1} \Rightarrow fg \in L^{p_2}$" is L^{p_3}, if $1/p_2 = 1/p_1 + 1/p_3$ with $p_i \in (0, \infty]$, $i = 1, 2, 3$. In this paper we characterize the set of all functions g such that "$f \in \mathrm{w}L^{\Phi_1} \Rightarrow fg \in \mathrm{w}L^{\Phi_2}$", where $\mathrm{w}L^{\Phi_i}$, $i = 1, 2$, are weak Orlicz spaces.

1. Introduction

Let $\Omega = (\Omega, \mu)$ be a complete σ-finite measure space. We denote by $L^0(\Omega)$ the set of all measurable functions from Ω to $\mathbb{R}$ or $\mathbb{C}$. Then $L^0(\Omega)$ is a linear space under the usual sum and scalar multiplication. Let $E_1, E_2 \subset L^0(\Omega)$ be subspaces. We say that a function $g \in L^0(\Omega)$ is a pointwise multiplier from E_1 to E_2, if the pointwise multiplication fg is in E_2 for any $f \in E_1$. We denote by $\mathrm{PWM}(E_1, E_2)$ the set of all pointwise multipliers from E_1 to E_2. We abbreviate $\mathrm{PWM}(E, E)$ to $\mathrm{PWM}(E)$. For example,

$$\mathrm{PWM}(L^0(\Omega)) = L^0(\Omega).$$

The pointwise multipliers are basic operators on function spaces and thus the characterization of pointwise multipliers is not only interesting itself but also sometimes very useful to other study.

For $p \in (0, \infty]$, $L^p(\Omega)$ denotes the usual Lebesgue space equipped with the norm

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, d\mu(x) \right)^{1/p}, \qquad \text{if } p \neq \infty,$$

$$\|f\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |f(x)|.$$

Then $L^p(\Omega)$ is a complete quasi-normed space (quasi-Banach space). If $p \in [1, \infty]$, then it is a Banach space. It is well known as Hölder's

---

The second author was supported by Grant-in-Aid for Scientific Research (B), No. 15H03621, Japan Society for the Promotion of Science.

2010 *Mathematics Subject Classification*. Primary 46E30.

Key words and phrases. Orlicz space, weak Orlicz space, pointwise multiplier.

inequality that

$$\|fg\|_{L^{p_2}(\Omega)} \le \|f\|_{L^{p_1}(\Omega)} \|g\|_{L^{p_3}(\Omega)},$$

for $1/p_2 = 1/p_1 + 1/p_3$ with $p_i \in (0, \infty]$, $i = 1, 2, 3$. This shows that

$$\mathrm{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega)) \supset L^{p_3}(\Omega),$$

and

$$\|g\|_{\mathrm{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega))} \le \|g\|_{L^{p_3}(\Omega)},$$

where $\|g\|_{\mathrm{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega))}$ is the operator norm of $g \in \mathrm{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega))$. Conversely, we can show the reverse inclusion by using the uniform boundedness theorem or the closed graph theorem. That is,

$$\mathrm{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega)) = L^{p_3}(\Omega) \quad \text{and} \quad \|g\|_{\mathrm{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega))} = \|g\|_{L^{p_3}(\Omega)}. \tag{1.1}$$

If $p_1 = p_2 = p$, then

$$\mathrm{PWM}(L^{p}(\Omega)) = L^{\infty}(\Omega) \qquad \text{and} \qquad \|g\|_{\mathrm{PWM}(L^{p}(\Omega))} = \|g\|_{L^{\infty}(\Omega)}. \tag{1.2}$$

Proofs of (1.1) and (1.2) are in Maligranda and Persson [12, Proposition 3 and Theorem 1]. See also [17] for a survey on pointwise multipliers. The characterization (1.1) was extended to several function spaces, for example, Orlicz spaces, Lorentz spaces, Morrey spaces, etc, see [1, 6, 7, 9, 11, 12, 13, 14, 15, 16, 18] and the references in [17].

In this paper we give the characterization of pointwise multipliers on weak Orlicz spaces. To do this we first prove a generalized Hölder's inequality for the weak Orlicz spaces. Next, to characterize the pointwise multipliers, we use the fact that all pointwise multipliers from a weak Orlicz space to another weak Orlicz space are bounded operators. This fact follows from Theorem 1.1 and Corollary 1.2 bellow.

We always assume that the function spaces $E \subset L^0(\Omega)$ have the following property, see [3, pages 94] in which this property is referred to as supp $E = \Omega$:

If a measurable subset $\Omega_1 \subset \Omega$ satisfies that

$$\mu(\{x \in \Omega : f(x) \neq 0\} \backslash \Omega_1) = 0 \text{ for every } f \in E,$$

$$\text{then } \mu(\Omega \backslash \Omega_1) = 0. \tag{1.3}$$

We say that a quasi-normed space $E \subset L^0(\Omega)$ has the lattice property if the following holds:

$$f \in E, \quad h \in L^0(\Omega), \quad |h| \le |f| \text{ a.e.} \quad \Rightarrow \quad h \in E, \quad \|h\|_E \le \|f\|_E. \tag{1.4}$$

Then we have the following theorem:

Theorem 1.1 ([17, Theorem 2.7]). *Let a quasi-normed space* $E \subset L^0(\Omega)$ *have the lattice property* (1.4). *For any sequence of functions* $f_j \in E$, $j = 1, 2, \dots$, *if* $f_j \to 0$ *in* E, *then* $f_j \to 0$ *in measure on every measurable set with finite measure.*

Using the closed graph theorem, we have the following corollary:

Corollary 1.2 ([17, Corollary 2.8]). *If* E_1 *and* E_2 *are complete quasi-normed spaces with the lattice property* (1.4), *then all* $g \in \mathrm{PWM}(E_1, E_2)$ *are bounded operators.*

Since the weak Orlicz spaces are complete quasi-normed spaces with the lattice property (1.4), all pointwise multipliers from a weak Orlicz space to another weak Orlicz space are bounded operators.

Orlicz spaces are introduced by [20, 21]. For the theory of Orlicz spaces, see [4, 5, 8, 10, 22] for example. See also [2] for the weak Orlicz space.

The organization of this paper is as follows. We recall the definitions of the Young functions and the weak Orlicz spaces in Section 2. Then we state main results in Section 3. The proof method is the same as [11]. However we need to investigate the properties of the quasi-norm on the weak Orlicz space. We do this in Section 4 to prove the main results in Section 5.

2. Young functions and weak Orlicz spaces

For an increasing function $\Phi : [0, \infty] \to [0, \infty]$, let

$$a(\Phi) = \sup\{t \ge 0 : \Phi(t) = 0\}, \qquad b(\Phi) = \inf\{t \ge 0 : \Phi(t) = \infty\},$$

with convention $\sup \varnothing = 0$ and $\inf \varnothing = \infty$. Then $0 \le a(\Phi) \le b(\Phi) \le \infty$.

Definition 2.1 (Young function). An increasing function $\Phi : [0, \infty] \to [0, \infty]$ is called a Young function (or sometimes also called an Orlicz function) if it satisfies the following properties;

$$0 \le a(\Phi) < \infty, \qquad 0 < b(\Phi) \le \infty, \tag{2.1}$$

$$\lim_{t \to +0} \Phi(t) = \Phi(0) = 0, \tag{2.2}$$

$$\Phi \text{ is convex on } [0, b(\Phi)), \tag{2.3}$$

$$\text{if } b(\Phi) = \infty, \text{ then } \lim_{t \to \infty} \Phi(t) = \Phi(\infty) = \infty, \tag{2.4}$$

$$\text{if } b(\Phi) < \infty, \text{ then } \lim_{t \to b(\Phi) - 0} \Phi(t) = \Phi(b(\Phi)) \ (\le \infty). \tag{2.5}$$

In what follows, if an increasing and convex function $\Phi : [0, \infty) \to [0, \infty)$ satisfies (2.2) and $\lim_{t\to\infty} \Phi(t) = \infty$, then we always regard that $\Phi(\infty) = \infty$ and that Φ is a Young function.

We denote by Φ_Y the set of all Young functions. We also define three subsets $\mathscr{Y}^{(i)}$ $(i = 1, 2, 3)$ of Φ_Y as

$$\mathscr{Y}^{(1)} = \{\Phi \in \Phi_Y : b(\Phi) = \infty\},$$

$$\mathscr{Y}^{(2)} = \{\Phi \in \Phi_Y : b(\Phi) < \infty,\ \Phi(b(\Phi)) = \infty\},$$

$$\mathscr{Y}^{(3)} = \{\Phi \in \Phi_Y : b(\Phi) < \infty,\ \Phi(b(\Phi)) < \infty\}.$$

REMARK 2.1. We have the following properties of $\Phi \in \Phi_Y$:

(Y1) If $\Phi \in \mathscr{Y}^{(1)}$, then Φ is absolutely continuous on any closed interval in $[0, \infty)$, and Φ is bijective from $[a(\Phi), \infty)$ to $[0, \infty)$.

(Y2) If $\Phi \in \mathscr{Y}^{(2)}$, then Φ is absolutely continuous on any closed interval in $[0, b(\Phi))$, and Φ is bijective from $[a(\Phi), b(\Phi))$ to $[0, \infty)$.

(Y3) If $\Phi \in \mathscr{Y}^{(3)}$, then Φ is absolutely continuous on $[0, b(\Phi)]$ and Φ is bijective from $[a(\Phi), b(\Phi)]$ to $[0, \Phi(b(\Phi))]$.

(Y4) If $\Phi \in \mathscr{Y}^{(3)}$ and $0 < \delta < 1$, then there exists a Young function $\Psi \in \mathscr{Y}^{(2)}$ such that $b(\Phi) = b(\Psi)$ and

$$\Psi(\delta t) \le \Phi(t) \le \Psi(t) \qquad \text{for all } t \in [0, \infty).$$

To see this we only set $\Psi = \Phi + \Theta$, where we choose $\Theta \in \mathscr{Y}^{(2)}$ such that $a(\Theta) = \delta b(\Phi)$ and $b(\Theta) = b(\Phi)$.

DEFINITION 2.2. For a Young function Φ, let

$$L^{\Phi}(\Omega) = \left\{ f \in L^0(\Omega) : \int_\Omega \Phi(\varepsilon|f(x)|)\,d\mu(x) < \infty \text{ for some } \varepsilon > 0 \right\},$$

$$\|f\|_{L^{\Phi}(\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1 \right\},$$

$$\mathrm{w}L^{\Phi}(\Omega) = \left\{ f \in L^0(\Omega) : \sup_{t\in(0,\infty)} \Phi(t)\mu(\varepsilon f, t) < \infty \text{ for some } \varepsilon > 0 \right\},$$

$$\|f\|_{\mathrm{w}L^{\Phi}(\Omega)} = \inf\left\{ \lambda > 0 : \sup_{t\in(0,\infty)} \Phi(t)\mu\left(\frac{f}{\lambda}, t\right) \le 1 \right\},$$

where $\mu(f, t) = \mu(\{x \in \Omega : |f(x)| > t\})$.

Then $\|\cdot\|_{L^{\Phi}(\Omega)}$ is a norm and thereby $L^{\Phi}(\Omega)$ is a Banach space, and $\|\cdot\|_{\mathrm{w}L^{\Phi}(\Omega)}$ is a quasi-norm and thereby $\mathrm{w}L^{\Phi}(\Omega)$ is a complete quasi-normed space (quasi-Banach space). For any Young function Φ,

$$L^{\Phi}(\Omega) \subset \mathrm{w}L^{\Phi}(\Omega) \qquad \text{with } \|f\|_{\mathrm{w}L^{\Phi}(\Omega)} \le \|f\|_{L^{\Phi}(\Omega)}.$$

Let

$$\Phi_{(\infty)}(t) = \begin{cases} 0, & t \in [0, 1], \\ \infty, & t \in (1, \infty]. \end{cases}$$

Then $\Phi_{(\infty)}$ is a Young function and

$$L^{\Phi_{(\infty)}}(\Omega) = \mathrm{w}L^{\Phi_{(\infty)}}(\Omega) = L^{\infty}(\Omega) \qquad \text{with}$$
$$\|f\|_{L^{\Phi_{(\infty)}}(\Omega)} = \|f\|_{\mathrm{w}L^{\Phi_{(\infty)}}(\Omega)} = \|f\|_{L^{\infty}(\Omega)}.$$

If Φ be a Young function with $b(\Phi) < \infty$, then $\Phi_{(\infty)}(t) \le \Phi(b(\Phi)t)$ for all $t \in [0, \infty]$. Hence,

$$\mathrm{w}L^{\Phi}(\Omega) \subset L^{\infty}(\Omega) \qquad \text{with } \|f\|_{L^{\infty}(\Omega)} \le b(\Phi)\|f\|_{\mathrm{w}L^{\Phi}(\Omega)}.$$

We note that

$$\sup_{t \in (0,\infty)} t\mu(\Phi(|f|), t) = \sup_{t \in (0,\infty)} \Phi(t)\mu(f, t), \tag{2.6}$$

and then

$$\begin{aligned} \|f\|_{\mathrm{w}L^{\Phi}(\Omega)} &= \inf\left\{\lambda > 0 : \sup_{t \in (0,\infty)} \Phi(t)\mu\left(\frac{f}{\lambda}, t\right) \le 1\right\} \\ &= \inf\left\{\lambda > 0 : \sup_{t \in (0,\infty)} t\mu\left(\Phi\left(\frac{|f|}{\lambda}\right), t\right) \le 1\right\}. \end{aligned}$$

We give a proof of (2.6) for readers' convenience, see Proposition 4.2.

Next we recall the generalized inverse of Young function Φ in the sense of O'Neil [19, Definition 1.2].

Definition 2.3. For a Young function Φ, let

$$\Phi^{-1}(u) = \begin{cases} \inf\{t \ge 0 : \Phi(t) > u\}, & u \in [0, \infty), \\ \infty, & u = \infty. \end{cases} \tag{2.7}$$

Then $\Phi^{-1}(u)$ is finite for all $u \in [0, \infty)$, continuous on $(0, \infty)$ and right continuous at $u = 0$. If Φ is bijective from $[0, \infty]$ to itself, then Φ^{-1} is the usual inverse function of Φ.

Remark 2.2. We have the following properties of $\Phi \in \Phi_Y$ and its inverse:

(P1) $\Phi(\Phi^{-1}(t)) \le t \le \Phi^{-1}(\Phi(t))$ for all $t \in [0, \infty]$ (Property 1.3 in [19]).
(P2) $\Phi^{-1}(\Phi(t)) = t$ if $\Phi(t) \in (0, \infty)$.
(P3) If $\Phi \in \mathscr{Y}^{(1)} \cup \mathscr{Y}^{(2)}$, then $\Phi(\Phi^{-1}(u)) = u$ for all $u \in [0, \infty]$.

Remark 2.3. Sometimes one defines

$$\Phi^{-1}(u) = \inf\{t \ge 0 : \Phi(t) > u\} \quad (u \in [0, \infty)) \qquad \text{and}$$

$$\Phi^{-1}(\infty) = \lim_{u \to \infty} \Phi^{-1}(u).$$

In this case $\Phi(\Phi^{-1}(u)) \le u$ for all $u \in [0, \infty)$ and $t \le \Phi^{-1}(\Phi(t))$ if $\Phi(t) \in [0, \infty)$.

3. Main results

For Young functions Φ_1 and Φ_2, we denote by $\|g\|_{\mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega), \mathrm{w}L^{\Phi_2}(\Omega))}$ the operator norm of $g \in \mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega), \mathrm{w}L^{\Phi_2}(\Omega))$. The following result is a generalized Hölder's inequality for the weak Orlicz spaces.

Theorem 3.1. *Let Φ_i, $i = 1, 2, 3$, be Young functions. If there exists a positive constant C such that, for all $u \in (0, \infty)$,*

$$\Phi_1^{-1}(u)\Phi_3^{-1}(u) \le C\Phi_2^{-1}(u), \tag{3.1}$$

then, for all $f \in \mathrm{w}L^{\Phi_1}(\Omega)$ and $g \in \mathrm{w}L^{\Phi_3}(\Omega)$,

$$\|fg\|_{\mathrm{w}L^{\Phi_2}(\Omega)} \le 4C\|f\|_{\mathrm{w}L^{\Phi_1}(\Omega)}\|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)}.$$

Consequently,

$$\mathrm{w}L^{\Phi_3}(\Omega) \subset \mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega), \mathrm{w}L^{\Phi_2}(\Omega)),$$

and, for all $g \in \mathrm{w}L^{\Phi_3}(\Omega)$,

$$\|g\|_{\mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega), \mathrm{w}L^{\Phi_2}(\Omega))} \le 4C\|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)}.$$

For the Orlicz spaces, it is known by O'Neil [19] that, if (3.1) holds, then

$$\|fg\|_{L^{\Phi_2}(\Omega)} \le 2C\|f\|_{L^{\Phi_1}(\Omega)}\|g\|_{L^{\Phi_3}(\Omega)}.$$

Next, we state our main result.

Theorem 3.2. *Let Φ_i, $i = 1, 2, 3$, be Young functions. If there exists a positive constant C such that, for all $u \in (0, \infty)$,*

$$\Phi_2^{-1}(u) \le C\Phi_1^{-1}(u)\Phi_3^{-1}(u), \tag{3.2}$$

then

$$\mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega), \mathrm{w}L^{\Phi_2}(\Omega)) \subset \mathrm{w}L^{\Phi_3}(\Omega),$$

and, for all $g \in \mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega), \mathrm{w}L^{\Phi_2}(\Omega))$,

$$\|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)} \leq C\|g\|_{\mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega), \mathrm{w}L^{\Phi_2}(\Omega))}.$$

In [11] it was shown that, if (3.2) holds, then

$$\mathrm{PWM}(L^{\Phi_1}(\Omega), L^{\Phi_2}(\Omega)) \subset L^{\Phi_3}(\Omega),$$

and, for all $g \in \mathrm{PWM}(L^{\Phi_1}(\Omega), L^{\Phi_2}(\Omega))$,

$$\|g\|_{L^{\Phi_3}(\Omega)} \leq C\|g\|_{\mathrm{PWM}(L^{\Phi_1}(\Omega), L^{\Phi_2}(\Omega))}.$$

Corollary 3.3. *Let* Φ_i, $i = 1, 2, 3$, *be Young functions. If there exist positive constants* C_i, $i = 1, 2$, *such that, for all* $u \in (0, \infty)$,

$$C_1^{-1}\Phi_2^{-1}(u) \leq \Phi_1^{-1}(u)\Phi_3^{-1}(u) \leq C_2\Phi_2^{-1}(u),$$

then

$$\mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega), \mathrm{w}L^{\Phi_2}(\Omega)) = \mathrm{w}L^{\Phi_3}(\Omega),$$

and

$$C_1^{-1}\|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)} \leq \|g\|_{\mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega), \mathrm{w}L^{\Phi_2}(\Omega))} \leq 4C_2\|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)}.$$

In the following, for functions $P, Q : [0, \infty) \to [0, \infty)$, $P(t) \sim Q(t)$ means that there exists a positive constant C such that $C^{-1}P(t) \leq Q(t) \leq CP(t)$ for all $t \in [0, \infty)$.

Example 3.1. Let $p_i, q_i \in [1, \infty)$, $i = 1, 2, 3$, and

$$\Phi_i(t) = t^{p_i}\max(1, \log t)^{q_i}, \qquad i = 1, 2, 3.$$

Then

$$\Phi_i^{-1}(t) \sim t^{1/p_i}\max(1, \log t)^{-q_i/p_i}, \qquad i = 1, 2, 3.$$

Hence, if $1/p_1 + 1/p_3 = 1/p_2$ and $q_1/p_1 + q_3/p_3 = q_2/p_2$, then

$$\mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega), \mathrm{w}L^{\Phi_2}(\Omega)) = \mathrm{w}L^{\Phi_3}(\Omega),$$

and the quasi-norms $\|\cdot\|_{\mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega), \mathrm{w}L^{\Phi_2}(\Omega))}$ and $\|\cdot\|_{\mathrm{w}L^{\Phi_3}(\Omega)}$ are equivalent.

Example 3.2. Let $p_i, q_i \in [1, \infty)$, $i = 1, 2, 3$, and

$$\Phi_i(t) = \exp(t^{p_i}) - 1, \qquad i = 1, 2, 3.$$

Then

$$\Phi_i^{-1}(t) \sim \begin{cases} t^{1/p_i}, & 0 \leq t < 2, \\ (\log t)^{1/p_i}, & 2 \leq t < \infty, \end{cases} \qquad i = 1, 2, 3.$$

Hence, if $1/p_1 + 1/p_3 = 1/p_2$, then

$$\mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega), \mathrm{w}L^{\Phi_2}(\Omega)) = \mathrm{w}L^{\Phi_3}(\Omega),$$

and the quasi-norms $\|\cdot\|_{\mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega), \mathrm{w}L^{\Phi_2}(\Omega))}$ and $\|\cdot\|_{\mathrm{w}L^{\Phi_3}(\Omega)}$ are equivalent.

4. Properties of the quasi-norm

In this section we investigate the properties of the quasi-norm $\|\cdot\|_{\mathrm{w}L^{\Phi}(\Omega)}$ to prove the main results.

For two Young functions Φ and Ψ, if there exist positive constants C_1 and C_2 such that

$$\Phi(C_1 t) \le \Psi(t) \le \Phi(C_2 t) \qquad \text{for all } t \in [0, \infty],$$

then $\mathrm{w}L^{\Phi}(\Omega) = \mathrm{w}L^{\Psi}(\Omega)$ and

$$C_1 \|f\|_{\mathrm{w}L^{\Phi}(\Omega)} \le \|f\|_{\mathrm{w}L^{\Psi}(\Omega)} \le C_2 \|f\|_{\mathrm{w}L^{\Phi}(\Omega)}.$$

By the measure theory we have the following property:

$$\begin{aligned} & f_j \ge 0 \quad \text{and} \quad f_j \nearrow f \ \text{ a.e.} \\ & \quad \Rightarrow \quad \lim_j \mu(f_j, t) = \mu(f, t) \quad \text{for each } t \in [0, \infty). \end{aligned} \tag{4.1}$$

From this property and the left continuity of the Young function Φ we have the following property:

$$\sup_{t \in (0,\infty)} t\mu\left(\Phi\left(\frac{|f(\cdot)|}{\|f\|_{\mathrm{w}L^{\Phi}(\Omega)}}\right), t\right) \le 1. \tag{4.2}$$

We also have the Fatou property:

$$\begin{aligned} & f_j \in \mathrm{w}L^{\Phi}(\Omega) \ (j = 1, 2 \cdots), \quad f_j \ge 0, \quad f_j \nearrow f \ \text{ a.e. and } \quad \sup_j \|f_j\|_{\mathrm{w}L^{\Phi}(\Omega)} < \infty, \\ & \quad \Rightarrow \quad f \in \mathrm{w}L^{\Phi}(\Omega) \quad \text{and} \quad \|f\|_{\mathrm{w}L^{\Phi}(\Omega)} \le \sup_j \|f_j\|_{\mathrm{w}L^{\Phi}(\Omega)}. \end{aligned} \tag{4.3}$$

PROPOSITION 4.1. *If $\Phi \in \mathscr{Y}^{(1)} \cup \mathscr{Y}^{(2)}$ and g is a finitely simple function and $g \ne 0$, then $g \in \mathrm{w}L^{\Phi}(\Omega)$ and*

$$\sup_{t \in (0,\infty)} t\mu\left(\Phi\left(\frac{|g(\cdot)|}{\|g\|_{\mathrm{w}L^{\Phi}(\Omega)}}\right), t\right) = 1.$$

PROOF. Let

$$I_{\Phi}(g) = \sup_{t \in (0,\infty)} t\mu(\Phi(|g(\cdot)|), t).$$

Case 1. $\Phi \in \mathscr{Y}^{(1)}$: In this case Φ is strictly increasing and bijective from $(a(\Phi), \infty)$ to $(0, \infty)$. Let g be a finitely simple function. We may assume that $g \geq 0$, i.e.,

$$g = \sum_{k=1}^{N} c_k \chi_{A_k}, \qquad 0 < c_1 < c_2 < \cdots < c_N < \infty,\ 0 < \mu(A_k) < \infty,$$

where A_k are pairwise disjoint. Then every $\Phi(c_k/\lambda)$ is continuous and decreasing with respect to $\lambda > 0$. Moreover, $\Phi(c_k/\lambda)$ is strictly decreasing on $(0, c_k/a(\Phi))$ (for $a(\Phi) = 0$ we understand $c_k/a(\Phi) = \infty$). Observing

$$\begin{aligned}\mu\left(\Phi\left(\frac{g}{\lambda}\right), t\right) &= \mu\left(\sum_{k=1}^{N} \Phi\left(\frac{c_k}{\lambda}\right)\chi_{A_k}, t\right) \\ &= \sum_{k=j}^{N} \mu(A_k), \qquad \text{if } \Phi\left(\frac{c_{j-1}}{\lambda}\right) \leq t < \Phi\left(\frac{c_j}{\lambda}\right), \quad j = 1, 2, \ldots, N,\end{aligned}$$

where $c_0 = 0$, we have

$$I_\Phi\left(\frac{g}{\lambda}\right) = \sup_{t \in (0,\infty)} t\mu\left(\Phi\left(\frac{g}{\lambda}\right), t\right) = \max_{1 \leq j \leq N} \Phi\left(\frac{c_j}{\lambda}\right) \sum_{k=j}^{N} \mu(A_k).$$

Therefore, $I_\Phi(g/\lambda)$ is continuous and strictly decreasing on $(0, c_N/a(\Phi))$. Since $\lim_{\lambda \to 0} I_\Phi(g/\lambda) = \infty$ and $\lim_{\lambda \to c_N/a(\Phi)} I_\Phi(g/\lambda) = 0$, we obtain that $I_\Phi(g/\cdot)$ is bijective from $(0, c_N/a(\Phi))$ to $(0, \infty)$. That is, there exists a unique $\lambda \in (0, c_N/a(\Phi))$ such that $I_\Phi(g/\lambda) = 1$.

Case 2. $\Phi \in \mathscr{Y}^{(2)}$: In this case Φ is strictly increasing and bijective from $(a(\Phi), b(\Phi))$ to $(0, \infty)$. Let g be a simple function as in Case 1. Then, in the same way as in Case 1, we obtain that $I_\Phi(g/\cdot)$ is bijective from $(c_N/b(\Phi), c_N/a(\Phi))$ to $(0, \infty)$. That is, there exists a unique $\lambda \in (c_N/b(\Phi), c_N/a(\Phi))$ such that $I_\Phi(g/\lambda) = 1$. □

In the rest of this section we show the following proposition.

Proposition 4.2. *For any Young function Φ,*

$$\sup_{t \in (0,\infty)} \Phi(t)\mu(f, t) = \sup_{u \in (0,\infty)} u\mu(f, \Phi^{-1}(u)) = \sup_{u \in (0,\infty)} u\mu(\Phi(|f(\cdot)|), u). \tag{4.4}$$

Remark 4.1. If $t = u = 0$, then

$$\Phi(t)\mu(f, t) = u\mu(f, \Phi^{-1}(u)) = u\mu(\Phi(|f(\cdot)|), u) = 0,$$

since $\Phi(0) = 0$. If $t = u = \infty$, then

$$\{x : |f(x)| > t\} = \{x : |f(x)| > \Phi^{-1}(u)\} = \{x : \Phi(|f(\cdot)|) > u\} = \varnothing,$$

since $\Phi^{-1}(\infty) = \infty$, that is,

$$\Phi(t)\mu(f, t) = u\mu(f, \Phi^{-1}(u)) = u\mu(\Phi(|f(\cdot)|), u) = 0.$$

LEMMA 4.3. *Let Φ be a Young function with $a(\Phi) < b(\Phi)$. If $u \in (0, \Phi(b(\Phi)))$, then*

$$\{x : |f(x)| > \Phi^{-1}(u)\} = \{x : \Phi(|f(x)|) > u\}.$$

PROOF. Let $a = a(\Phi)$ and $b = b(\Phi)$. Then Φ is bijective from (a, b) to $(0, \Phi(b))$ in any case of $b < \infty$ or $b = \infty$; $\Phi(b) < \infty$ or $\Phi(b) = \infty$. Let $t = \Phi^{-1}(u)$. Then

$$t \in (a, b) \Leftrightarrow u \in (0, \Phi(b)).$$

If $|f(x)| \in (a, b)$, then

$$|f(x)| > t \Leftrightarrow \Phi(|f(x)|) > \Phi(t).$$

That is,

$$|f(x)| > \Phi^{-1}(u) \Leftrightarrow \Phi(|f(x)|) > u.$$

If $|f(x)| \le a$, then

$$|f(x)| \le a < t = \Phi^{-1}(u) \quad \text{and} \quad \Phi(|f(x)|) = 0 < u.$$

If $|f(x)| \ge b$, then

$$|f(x)| \ge b > t = \Phi^{-1}(u) \quad \text{and} \quad \Phi(|f(x)|) \ge \Phi(b) > u.$$

Therefore, we have the conclusion. □

PROOF (Proof of Proposition 4.2). Let $a = a(\Phi)$ and $b = b(\Phi)$.

Case 1: Let $\Phi \in \mathscr{Y}^{(1)} \cup \mathscr{Y}^{(2)}$. Then Φ is bijective from (a, b) to $(0, \infty)$, and then

$$\begin{aligned}\sup_{t \in (0,b)} \Phi(t)\mu(f, t) &= \sup_{t \in (a,b)} \Phi(t)\mu(f, t) \\ &= \sup_{u \in (0,\infty)} u\mu(f, \Phi^{-1}(u)) \\ &= \sup_{u \in (0,\infty)} u\mu(\Phi(|f(\cdot)|), u),\end{aligned}$$

where we used Lemma 4.3 for the last equality. If $b = \infty$, then the above equalities show (4.4). If $b < \infty$, then

$$\sup_{t \in [b,\infty)} \Phi(t)\mu(f, t) = 0 \text{ or } \infty.$$

If $\sup_{t \in [b, \infty)} \Phi(t)\mu(f, t) = 0$, then

$$\sup_{t \in (0, \infty)} \Phi(t)\mu(f, t) = \sup_{t \in (0, b)} \Phi(t)\mu(f, t).$$

If $\sup_{t \in [b, \infty)} \Phi(t)\mu(f, t) = \infty$, then $\mu(f, b) > 0$ and $\lim_{t \to b-0} \Phi(t)\mu(f, t) = \infty$. Hence

$$\sup_{t \in (0, \infty)} \Phi(t)\mu(f, t) = \sup_{t \in (0, b)} \Phi(t)\mu(f, t) = \infty.$$

Therefore, we have (4.4).

Case 2: Let $\Phi \in \mathscr{Y}^{(3)}$ and $a < b$. Then Φ is bijective from (a, b) to $(0, \Phi(b))$, and then

$$\begin{aligned}\sup_{t \in (0, b)} \Phi(t)\mu(f, t) &= \sup_{t \in (a, b)} \Phi(t)\mu(f, t) \\ &= \sup_{u \in (0, \Phi(b))} u\mu(f, \Phi^{-1}(u)) \\ &= \sup_{u \in (0, \Phi(b))} u\mu(\Phi(|f(\cdot)|), u),\end{aligned}$$

where we used Lemma 4.3 for the last equality. If $\mu(f, b) = 0$, then

$$\mu(f, \Phi^{-1}(u)) = \mu(\Phi(|f(\cdot)|), u) = 0 \qquad \text{for } u \in [\Phi(b), \infty),$$

since $\Phi^{-1}(u) = b$ and

$$\{x : \Phi(|f(x)|) > u\} \subset \{x : \Phi(|f(x)|) > \Phi(b)\} \subset \{x : |f(x)| > b\}.$$

Hence,

$$\sup_{t \in [b, \infty)} \Phi(t)\mu(f, t) = \sup_{u \in [\Phi(b), \infty)} u\mu(f, \Phi^{-1}(u)) = \sup_{u \in [\Phi(b), \infty)} u\mu(\Phi(|f(\cdot)|), u) = 0.$$

If $\mu(f, b) > 0$, then $\mu(f, b + 1/j) > 0$ for some $j \in \mathbb{N}$ by the measure theory. Hence,

$$\sup_{t \in (0, \infty)} \Phi(t)\mu(f, t) \geq \Phi(b + 1/j)\mu(f, b + 1/j) = \infty.$$

On the other hand, $\mu(f, b) > 0$ implies that, for all $u \in (\Phi(b), \infty)$,

$$\mu(f, \Phi^{-1}(u)) = \mu(f, b) > 0, \qquad \mu(\Phi(|f(\cdot)|), u) \geq \mu(\{x : \Phi(|f(x)|) = \infty\}) > 0.$$

Hence,

$$\sup_{u \in (0, \infty)} u\mu(f, \Phi^{-1}(u)) = \sup_{u \in (0, \infty)} u\mu(\Phi(|f(\cdot)|), u) = \infty.$$

Therefore, we have (4.4).

Case 3: Let $\Phi \in \mathscr{Y}^{(3)}$ and $a = b$. Then $\Phi(t) = 0$ for $t \in (0, b]$ and $\Phi^{-1}(u) = b$ for $u \in (0, \infty)$. If $\mu(f, b) = 0$, then $|f(x)| \le b$ and $\Phi(|f(x)|) = 0$ a.e. x. Hence

$$\sup_{t \in (0, \infty)} \Phi(t)\mu(f, t) = \sup_{u \in (0, \infty)} u\mu(f, \Phi^{-1}(u)) = \sup_{u \in (0, \infty)} u\mu(\Phi(|f(\cdot)|), u) = 0.$$

If $\mu(f, b) > 0$, then, by the same way as Case 2, we have

$$\sup_{t \in (0, \infty)} \Phi(t)\mu(f, t) = \sup_{u \in (0, \infty)} u\mu(f, \Phi^{-1}(u)) = \sup_{u \in (0, \infty)} u\mu(\Phi(|f(\cdot)|), u) = \infty.$$

Therefore, we have (4.4). □

5. Proofs

Proof (Proof of Theorem 3.1). Let $f \in \mathrm{w}L^{\Phi_1}(\Omega)$ and $g \in \mathrm{w}L^{\Phi_3}(\Omega)$. We may assume that $f, g \ge 0$ and $\|f\|_{\mathrm{w}L^{\Phi_1}(\Omega)} = \|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)} = 1$. Let

$$h(x) = \max(\Phi_1(f(x)), \Phi_3(g(x))).$$

Then, by the assumption (3.1) and (P1),

$$\begin{aligned} f(x)g(x) &\le \Phi_1^{-1}(\Phi_1(f(x)))\Phi_3^{-1}(\Phi_3(g(x))) \\ &\le \Phi_1^{-1}(h(x))\Phi_3^{-1}(h(x)) \le C\Phi_2^{-1}(h(x)). \end{aligned}$$

Hence, by (P1),

$$\Phi_2\left(\frac{f(x)g(x)}{C}\right) \le \Phi_2(\Phi_2^{-1}(h(x))) \le h(x) \le \Phi_1(f(x)) + \Phi_3(g(x)).$$

Then

$$\begin{aligned} &\sup_{t \in (0, \infty)} t\mu\left(\Phi_2\left(\frac{f(x)g(x)}{4C}\right), t\right) \\ &\le \sup_{t \in (0, \infty)} t\mu\left(\frac{1}{4}\Phi_2\left(\frac{f(x)g(x)}{C}\right), t\right) \\ &= \frac{1}{2} \sup_{t \in (0, \infty)} t\mu\left(\Phi_2\left(\frac{f(x)g(x)}{C}\right), 2t\right) \\ &\le \frac{1}{2} \sup_{t \in (0, \infty)} t\mu(\Phi_1(f(x)) + \Phi_3(g(x)), 2t) \end{aligned}$$

$$\leq \frac{1}{2} \sup_{t \in (0,\infty)} t(\mu(\Phi_1(f(x)), t) + \mu(\Phi_3(g(x)), t))$$

$$\leq \frac{1}{2}(1+1) = 1,$$

where we used (4.2) for the last inequality. Therefore, $\|fg\|_{wL^{\Phi_2}(\Omega)} \leq 4C$ and the proof is complete. □

Proof (Proof of Theorem 3.2). **Case 1.** Φ_2 and Φ_3 are in $\mathscr{Y}^{(1)} \cup \mathscr{Y}^{(2)}$: Let

$$g \in \mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega), \mathrm{w}L^{\Phi_2}(\Omega)).$$

Assume first that g is a finitely simple function. Then $g \in \mathrm{w}L^{\Phi_3}(\Omega)$ and

$$G(x) := \Phi_3\left(\frac{|g(x)|}{\|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)}}\right) < \infty \qquad \text{a.e. in } \Omega.$$

Put

$$h(x) = \begin{cases} \Phi_1^{-1}(G(x)), & \text{if } 0 < G(x) < \infty, \\ 0, & \text{if } G(x) = 0. \end{cases}$$

From the property (P1) it follows that $\Phi_1(h(x)) \leq G(x)$ a.e. in Ω and

$$\sup_{t \in (0,\infty)} t\mu(\Phi_1(h), t) \leq \sup_{t \in (0,\infty)} t\mu\left(\Phi_3\left(\frac{|g(\cdot)|}{\|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)}}\right), t\right) \leq 1,$$

which gives $\|h\|_{\mathrm{w}L^{\Phi_1}(\Omega)} \leq 1$. Next we show that

$$\Phi_2\left(Ch(x)\frac{|g(x)|}{\|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)}}\right) \geq G(x) = \Phi_3\left(\frac{|g(x)|}{\|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)}}\right). \tag{5.1}$$

If $0 < G(x) < \infty$, then by the property (P2) and the assumption (3.2),

$$h(x)\frac{|g(x)|}{\|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)}} = \Phi_1^{-1}(G(x))\Phi_3^{-1}\left(\Phi_3\left(\frac{|g(x)|}{\|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)}}\right)\right)$$

$$= \Phi_1^{-1}(G(x))\Phi_3^{-1}(G(x)) \geq \frac{1}{C}\Phi_2^{-1}(G(x))$$

and hence, by (P3),

$$\Phi_2\left(Ch(x)\frac{|g(x)|}{\|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)}}\right) \geq \Phi_2(\Phi_2^{-1}(G(x))) = G(x).$$

If $G(x) = 0$, then $h(x) = 0$ and $\Phi_2\left(Ch(x)\frac{|g(x)|}{\|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)}}\right) = 0$. Thus, we have (5.1). By Proposition 4.1 we have

$$\sup_{t\in(0,\infty)} t\mu\left(\Phi_2\left(Ch(\cdot)\frac{|g(\cdot)|}{\|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)}}\right), t\right) \ge \sup_{t\in(0,\infty)} t\mu\left(\Phi_3\left(\frac{|g(\cdot)|}{\|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)}}\right), t\right) = 1$$

and so $\|hg\|_{\mathrm{w}L^{\Phi_2}(\Omega)} \ge \frac{1}{C}\|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)}$, that is,

$$\|g\|_{\mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega),\mathrm{w}L^{\Phi_2}(\Omega))} \ge \frac{1}{C}\|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)},$$

where we use the fact that the pointwise multiplier g is a bounded operator.

In the general case, g can be approximated by a sequence of finitely simple functions $\{g_j\}$ such that $0 \le g_j \nearrow |g|$ a.e. in Ω, since μ is a σ-finite measure. Then

$$\|g\|_{\mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega),\mathrm{w}L^{\Phi_2}(\Omega))} \ge \|g_j\|_{\mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega),\mathrm{w}L^{\Phi_2}(\Omega))} \ge \frac{1}{C}\|g_j\|_{\mathrm{w}L^{\Phi_3}(\Omega)}$$

by our first part of the proof. Using the Fatou property (4.3) of the quasi-norm $\|\cdot\|_{\mathrm{w}L^{\Phi_3}(\Omega)}$, we obtain

$$\|g\|_{\mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega),\mathrm{w}L^{\Phi_2}(\Omega))} \ge \frac{1}{C}\|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)}.$$

Case 2. $\Phi_2 \in \mathscr{Y}^{(3)}$ or $\Phi_3 \in \mathscr{Y}^{(3)}$: We consider only the case that both Φ_2 and Φ_3 are in $\mathscr{Y}^{(3)}$, since other cases are similar. In this case, by (Y4), for all $0 < \delta < 1$, there exist $\Psi_2 \in \mathscr{Y}^{(2)}$ and $\Psi_3 \in \mathscr{Y}^{(2)}$ such that

$$\Psi_2(\delta u) \le \Phi_2(u) \le \Psi_2(u), \qquad \Psi_3(\delta u) \le \Phi_3(u) \le \Psi_3(u) \qquad \text{for all } u.$$

It follows that

$$\delta\Phi_2^{-1}(u) \le \Psi_2^{-1}(u) \le \Phi_2^{-1}(u), \qquad \delta\Phi_3^{-1}(u) \le \Psi_3^{-1}(u) \le \Phi_3^{-1}(u),$$

$$\delta\|g\|_{\mathrm{w}L^{\Psi_2}(\Omega)} \le \|g\|_{\mathrm{w}L^{\Phi_2}(\Omega)} \le \|g\|_{\mathrm{w}L^{\Psi_2}(\Omega)},$$

$$\delta\|g\|_{\mathrm{w}L^{\Psi_3}(\Omega)} \le \|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)} \le \|x\|_{\mathrm{w}L^{\Psi_3}(\Omega)},$$

and

$$\delta\|g\|_{\mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega),\mathrm{w}L^{\Psi_2}(\Omega))} \le \|g\|_{\mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega),\mathrm{w}L^{\Phi_2}(\Omega))} \le \|g\|_{\mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega),\mathrm{w}L^{\Psi_2}(\Omega))}.$$

Using the inequality

$$\Psi_2^{-1}(u) \le \frac{C}{\delta}\Phi_1^{-1}(u)\Psi_3^{-1}(u),$$

which follows by (3.2) and the definitions of Ψ_2 and Ψ_3, we have

$$\|g\|_{\mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega),\mathrm{w}L^{\Psi_2}(\Omega))} \geq \frac{\delta}{C}\|g\|_{\mathrm{w}L^{\Psi_3}(\Omega)}$$

by Case 1. Then

$$\|g\|_{\mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega),\mathrm{w}L^{\Phi_2}(\Omega))} \geq \frac{\delta^2}{C}\|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)}$$

holds for all $0 < \delta < 1$. Therefore,

$$\|g\|_{\mathrm{PWM}(\mathrm{w}L^{\Phi_1}(\Omega),\mathrm{w}L^{\Phi_2}(\Omega))} \geq \frac{1}{C}\|g\|_{\mathrm{w}L^{\Phi_3}(\Omega)},$$

and the proof is finished. □

Acknowledgement

The authors would like to thank Professors Hiro-o Kita and Takashi Miyamoto for their useful comments. The authors also would like to thank the referee for her/his useful comments.

References

[1] J. M. Calabuig, O. Delgado and E. A. Sánchez Pérez, Generalized perfect spaces, Indag. Math. (N.S.) 19 (2008), no. 3, 359–378.
[2] B. Iaffei, Comparison of two weak versions of the Orlicz spaces, Rev. Un. Mat. Argentina 40 (1996), no. 1–2, 191–202.
[3] L. V. Kantorovich and G. P. Akilov, Functional analysis, Translated from the Russian by Howard L. Silcock, Second edition, Pergamon Press, Oxford-Elmsford, N.Y., 1982.
[4] H. Kita, Orlicz spaces and their applications (Japanese), Iwanami Shoten, Publishers, Tokyo, 2009.
[5] V. Kokilashvili and M. Krbec, Weighted inequalities in Lorentz and Orlicz spaces, World Scientific Publishing Co., Inc., River Edge, NJ, 1991.
[6] P. Kolwicz, K. Leśnik and L. Maligranda, Pointwise multipliers of Calderón-Lozanovskiĭ spaces, Math. Nachr. 286 (2013), no. 8–9, 876–907.
[7] P. Kolwicz, K. Leśnik and L. Maligranda, Pointwise products of some Banach function spaces and factorization, J. Funct. Anal. 266 (2014), no. 2, 616–659.
[8] M. A. Krasnoselsky and Y. B. Rutitsky, Convex functions and Orlicz spaces, Translated from the first Russian edition by Leo F. Boron, P. Noordhoff Ltd., Groningen 1961.
[9] K. Lesnik and J. Tomaszewski, Pointwise multipliers of Orlicz function spaces and factorization, Positivity 21 (2017), no. 4, 1563–1573.
[10] L. Maligranda, Orlicz Spaces and Interpolation, Seminars in Math. Vol. 5, Univ. of Campinas, Campinas SP, Brazil, 1989.
[11] L. Maligranda and E. Nakai, Pointwise multipliers of Orlicz spaces, Arch. Math. (Basel) 95 (2010), no. 3, 251–256.

[12] L. Maligranda and L. E. Persson, Generalized duality of some Banach function spaces, Indag. Math. 51 (1989), no. 3, 323–338.
[13] E. Nakai, Pointwise multipliers on the Lorentz spaces, Mem. Osaka Kyoiku Univ. III Natur. Sci. Appl. Sci. 45 (1996), no. 1, 1–7.
[14] E. Nakai, Pointwise multipliers on the Morrey spaces, Mem. Osaka Kyoiku Univ. III Natur. Sci. Appl. Sci. 46 (1997), 1–11.
[15] E. Nakai, A characterization of pointwise multipliers on the Morrey spaces, Sci. Math. 3 (2000), no. 3, 445–454.
[16] E. Nakai, Pointwise multipliers on Musielak-Orlicz spaces, Nihonkai Math. J. 27 (2016), 135–146.
[17] E. Nakai, Pointwise multipliers on several functions spaces—a survey, Linear Nonlinear Anal. 3 (2017), no. 1, 27–59.
[18] E. Nakai, Pointwise multipliers on Musielak-Orlicz-Morrey spaces. Function spaces and inequalities, 257–281, Springer Proc. Math. Stat., 206, Springer, Singapore, 2017.
[19] R. O'Neil, Fractional integration in Orlicz spaces. I., Trans. Amer. Math. Soc. 115 (1965), 300–328.
[20] W. Orlicz, Über eine gewisse Klasse von Räumen vom Typus B, Bull. Acad. Polonaise A (1932), 207–220; reprinted in his Collected Papers, PWN, Warszawa 1988, 217–230.
[21] W. Orlicz, Über Räume (L^M), Bull. Acad. Polonaise A (1936), 93–107; reprinted in his Collected Papers, PWN, Warszawa 1988, 345–359.
[22] M. M. Rao and Z. D. Ren, Theory of Orlicz Spaces, Marcel Dekker, Inc., New York, Basel and Hong Kong, 1991.

Ryota Kawasumi
Department of Intelligent Systems Engineering
Ibaraki University
Hitachi, Ibaraki 316-8511*, Japan*
E-mail: rykawasumi@gmail.com

Eiichi Nakai
Department of Mathematics
Ibaraki University
Mito, Ibaraki 310-8512*, Japan*
E-mail: eiichi.nakai.math@vc.ibaraki.ac.jp