HIROSHIMA MATH. J.
50 (2020), 43–57

Twisted Alexander polynomials of $(-2, 3, 2n+1)$-pretzel knots

Airi Aso

(Received December 4, 2018)
(Revised August 31, 2019)

Abstract. We calculate the twisted Alexander polynomials of $(-2, 3, 2n+1)$-pretzel knots associated to the family of their $SL_2(\mathbb{C})$-representations which contains their holonomy representations.

1. Introduction

1.1. Motivations. The twisted Alexander polynomial is a generalization of the Alexander polynomial, and it is defined for the pair of a group and its representations. The notion of twisted Alexander polynomials was introduced by Wada [W] and Lin [L] independently in 1990s. The definition of Lin is for knots in S^3 and the definition of Wada is for finitely presented groups. By Kitano and Morifuji [KM], it is known that Wada's twisted Alexander polynomials of the knot groups for any nonabelian representations into $SL_2(\mathbb{F})$ over a field $\mathbb{F}$ are polynomials. As a corollary of the claim, they also showed that if K is a fibered knot of genus g, then its twisted Alexander polynomials are monic polynomials of degree $4g-2$ for any nonabelian $SL_2(\mathbb{F})$-representations. They also showed that there exists a nonfibered knot which has an $SL_2(\mathbb{C})$-representation such that the twisted Alexander polynomial of the representation is monic (see [GoMo]).

If K is hyperbolic, i.e. the complement $S^3 \backslash K$ admits a complete hyperbolic metric of finite volume, the most important representation is its holonomy representation into $SL_2(\mathbb{C})$ which is a lift of the representation into the group of orientation-preserving isometries of the hyperbolic 3-space $\mathbb{H}^3$. Dunfield, Friedl and Jackson [DFJ] conjectured that the twisted Alexander polynomials of hyperbolic knots associated to their holonomy representations (so-called hyperbolic torsion polynomials) determine the genus and fiberedness of the knots. In fact, they computed the twisted Alexander polynomials of all hyperbolic knots of 15 or fewer crossings associated to their holonomy representations, and the conjecture is verified for these hyperbolic knots. Recently, the twisted Alexander polynomials of some infinite families of knots, twist knots

---

2010 *Mathematics Subject Classification*. Primary 57M27; Secondary 57M25.
Key words and phrases. twisted Alexander polynomials, pretzel knot, holonomy representation.

and genus one two-bridge knots associated to their holonomy representations, are computed by Morifuji [Mo1] and Tran [T1], and genus one two-bridge knots associated to the adjoint representations of their holonomy representations is also computed by Tran [T2]. These examples are also supporting evidences of the conjecture. Dunfield, Friedl and Jackson also observed that the second highest coefficients of the hyperbolic torsion polynomials are often real for fibered knots however they are not very often real for non-fibered knots.

In this paper, we compute the twisted Alexander polynomials of $(-2, 3, 2n+1)$-pretzel knots K_n depicted in Figure 1 associated to the family of their $SL_2(\mathbb{C})$-representations which contains their holonomy representations given in the following section where integer n is not 0, 1, 2 ($(-2, 3, 2n+1)$-pretzel knots are hyperbolic knots for $n \neq 0, 1, 2$). The twisted Alexander polynomials of K_n are monic polynomials of degree $4(|n+1|+1)-2$ where $n \leq -2$ or $2 < n$, and the twisted Alexander polynomial of K_{-1} is a non-monic polynomial of degree 2. We can observe that K_n is fibered for integers $n \neq -1$ and the genus of K_n is $|n+1|+1$ (see [HM, Mu, O] for more details). Hence Dunfield, Friedl and Jackson's conjecture holds for $(-2, 3, 2n+1)$-pretzel knots. Furthermore, the second highest coefficients of the twisted Alexander polynomials of K_n associated to their holonomy representations are 0 for $n > 2$ and are -2 for $n \leq -2$, i.e. the second highest coefficients are real when K_n is fibered. This result coincide with the question of Dunfield, Friedl and Jackson. In contrast, the second highest coefficient of the twisted Alexander polynomial of K_{-1} associated to the holonomy representation is also real. This is a rare case for non-fibered knots.

On the other hand, $(-2, 3, 2n+1)$-pretzel knot is an infinite family of knots which contains the Fintushel-Stern knot i.e. $(-2, 3, 7)$-pretzel knot. It plays an important role in studying of exceptional surgeries of knots [Ma]. In fact, the A-polynomials of $(-2, 3, 2n+1)$-pretzel knot are computed by Tamura-Yokota [TY] and Garoufalidis-Mattman [GaMa]. We hope this result

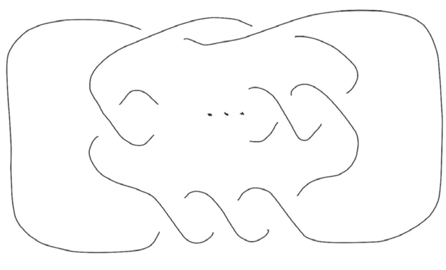

Fig. 1. $(-2, 3, 2n+1)$-pretzel knot

will be used as examples of the twisted Alexander polynomials of hyperbolic knots for solving some questions and open problems.

1.2. Definition of twisted Alexander polynomials. In this paper, we use the following definition due to Wada.

Definition 1. Let $G(K) = \pi_1(S^3 \backslash K)$ be the knot group of a knot K. The existence of a presentation of the form

$$G(K) = \langle x_1, \dots, x_n \,|\, r_1, \dots, r_{n-1} \rangle$$

is well known for any knots (see [CF]). Let Γ denote the free group generated by $x_1, \dots, x_n$ and $\phi : \mathbb{Z}\Gamma \to \mathbb{Z}G(K)$ the natural ring homomorphism induced by the presentation of $G(K)$. Let $\rho : G(K) \to GL_d(\mathbb{F})$ be a d-dimensional linear representation of $G(K)$ and $\Phi : \mathbb{Z}\Gamma \to M_d(\mathbb{F}[t, t^{-1}])$ the ring homomorphism defind by

$$\Phi = (\tilde{\rho} \otimes \tilde{\alpha}) \circ \phi,$$

where $\tilde{\alpha} : \mathbb{Z}G(K) \to \mathbb{Z}\langle t, t^{-1} \rangle$ and $\tilde{\rho}$ are respective ring homomorphisms induced by the abelianization $\alpha : G(K) \to \langle t \rangle$ and ρ. We put

$$A_{i,j} = \Phi\left(\frac{\partial r_i}{\partial x_j}\right),$$

where $\dfrac{\partial}{\partial x_j}$ denotes the Fox derivative (or free derivative) with respect to x_j, that is, a map $\mathbb{Z}\Gamma \to \mathbb{Z}\Gamma$ satisfying the conditions

$$\frac{\partial}{\partial x_j} x_i = \delta_{ij} \qquad \text{and} \qquad \frac{\partial}{\partial x_j} gg' = \frac{\partial}{\partial x_j} g + g \frac{\partial}{\partial x_j} g',$$

where δ_{ij} denotes the Kronecker symbol and $g, g' \in \Gamma$. Then, the twisted Alexander polynomial of K is defined by

$$\Delta_{K,\rho} = \frac{\det A_{\rho,k}}{\det \Phi(x_k - 1)},$$

where $A_{\rho,k}$ is the $2(n-1) \times 2(n-1)$ matrix obtained from $A_\rho = (A_{i,j})$ by removing the k-th column, i.e.

$$A_{\rho,k} = \begin{pmatrix} A_{1,1} & \cdots & A_{1,k-1} & A_{1,k+1} & \cdots & A_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{n-1,1} & \cdots & A_{n-1,k-1} & A_{n-1,k+1} & \cdots & A_{n-1,n} \end{pmatrix}.$$

2. Presentations and holonomy representations

In this section, we give a presentation of the knot group $G(K_n)$ and its holonomy representation $\rho_m : G(K_n) \to SL_2(\mathbb{C})$, where m represents the eigenvalue of the image of the meridian of K_n.

Let L be the link depicted in Figure 2 and $E = S^3 \backslash L$. Then, the Wirtinger presentation (see [CF]) of $\pi_1(E)$ is given by

$$\langle a, b, x \mid \{axba(xb)^{-1}\}^{-1}x = xb\{axba(xb)^{-1}\}^{-1}(axb)^{-1}xb,\ [x, axba(xb)^{-1}] = 1\rangle,$$

where a, b and x are Wirtinger generators assigned to the corresponding paths depicted in Figure 2. Note that $E_n := S^3 \backslash K_n$ is obtained from L by $\left(-\frac{1}{n}\right)$-surgery along the trivial component, that is, removing the tubular neighborhood of the trivial component and re-gluing the solid torus again after twisting $-n$ times along the longitude. Therefore, by the van Kampen theorem, we have

$$\begin{aligned}\pi_1(E_n) = \langle a, b, x \mid &\{axba(xb)^{-1}\}^{-1}x = xb\{axba(xb)^{-1}\}^{-1}(axb)^{-1}xb,\\ &x = \{axba(xb)^{-1}\}^n\rangle.\end{aligned}$$

Proposition 1. *For a non-zero complex number m, there exists a representation $\rho_m : \pi_1(E_n) \to SL_2(\mathbb{C})$ such that*

$$\rho_m(a) = \begin{pmatrix} m & -\dfrac{(m^2 - s)(s^{2n+1} + 1)}{m(s+1)} \\ 0 & m^{-1} \end{pmatrix},$$

$$\rho_m(b) = \frac{1}{s\alpha}\begin{pmatrix} \beta & -\dfrac{(s\alpha - m\beta)(ms\alpha - \beta)}{m\beta} \\ \beta & \dfrac{m(ms\alpha - \beta) + s\alpha}{m} \end{pmatrix},$$

and

$$\rho_m(x) = \begin{pmatrix} s^n & 0 \\ \dfrac{s^n - s^{-n}}{s^{2n+1} + 1} & s^{-n} \end{pmatrix},$$

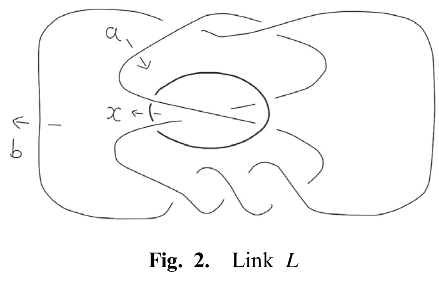

Fig. 2. Link L

where s is a solution of

$$\begin{aligned}0 &= m^8(s-1)(s+1)^2(s^{2n}-s^2)s^{2n+2}\\&\quad - m^6\{s^{6n+3}+(2s^6+s^5-4s^4+s^3+s^2-s-1)s^{4n+1}\\&\quad - (s^6+s^5-s^4-s^3+4s^2-s-2)s^{2n+2}+s^6\}\\&\quad + m^4\{(s^2+1)s^{6n+2}+(s^6+2s^5-3s^4-2s^3+6s^2-4s-2)s^{4n+3}\\&\quad - (2s^6+4s^5-6s^4+2s^3+3s^2-2s-1)s^{2n}+(s^2+1)s^5\}\\&\quad - m^2\{s^{6n+3}+(2s^6+s^5-4s^4+s^3+s^2-s-1)s^{4n+1}\\&\quad - (s^6+s^5-s^4-s^3+4s^2-s-2)s^{2n+2}+s^6\}\\&\quad + (s-1)(s+1)^2(s^{2n}-s^2)s^{2n+2} \qquad (1)\end{aligned}$$

and α, β are given by

$$\begin{aligned}\alpha &= (s^2-1)s^{2n}\{-m^6(s-1)s^2(s^{2n+1}+1)+m^4(s^{2n+2}(s^4-2s^2+3s-1)\\&\quad + s^4-3s^3+2s^2-1)-m^2s(s^{2n}(2s^3-s^2+1)\\&\quad - s(s^3-s+2))+s^2(s^{2n}-s^2)\},\\ \beta &= m^7s^{2n+2}(s^2-1)(s^3+1)-m^5s^3\{s^{4n}(s^3-s^2+1)\\&\quad + s^{2n-2}(s-1)(s^3+s+1)(s^3+s^2+1)-(s^3-s+1)\}\\&\quad + m^3s^2(s^3+1)(s^{2n}-1)(s^{2n}+s^2)-ms^3(s^{2n}-s^2)(s^{2n}+s).\end{aligned}$$

In what follows, for simplicity, we denote the right hand side of (1) by r_0.

Proof. For simplicity, put $A=\rho_m(a)$, $B=\rho_m(b)$, $X=\rho_m(x)$. By the aid of Mathematica, we have

$$\begin{aligned}AXBA(XB)^{-1} &= \begin{pmatrix} s & 0\\ \dfrac{s^2-1}{s(s^{2n+1}+1)} & \dfrac{1}{s}\end{pmatrix}\\&\quad + \frac{r_1}{m^3s\alpha^2}\begin{pmatrix} \dfrac{1}{s^{2n+1}+1} & -\dfrac{1}{s+1}\\ \dfrac{s+1}{s(s^{2n+1}+1)^2} & -\dfrac{1}{s(s^{2n+1}+1)}\end{pmatrix},\end{aligned}$$

where

$$r_1 = -\alpha^2 ms(m^2 s^{2n+2} - m^2 - s^{2n+1} + s) + \alpha\beta(m^2-1)(m^2+1)s^{2n+1}(s+1)$$
$$+ \beta^2 ms^{2n}(m^2 s^{2n+1} - m^2 s - s^{2n+2} + 1) \equiv 0 \bmod r_0.$$

Therefore, by (1), we have $X = \{AXBA(XB)^{-1}\}^n$, that is, $\rho_m(x)$ is equal to $\rho_m(\{axba(xb)^{-1}\}^n)$.

On the other hand, we can observe

$$AXB\{AXBA(XB)^{-1}\} \equiv XBX^{-1}\{AXBA(XB)^{-1}\}XB \bmod r_0$$

and so $AXB\{AXBA(XB)^{-1}\} = XBX^{-1}\{AXBA(XB)^{-1}\}XB$ by (1). Further more, we obtain

$$\begin{aligned} &XB\{AXBA(XB)^{-1}\}^{-1}(AXB)^{-1}XB \\ &\quad = XB(AXB\{AXBA(XB)^{-1}\})^{-1}XB \\ &\quad = XB(XBX^{-1}\{AXBA(XB)^{-1}\}XB)^{-1}XB \\ &\quad = \{AXBA(XB)^{-1}\}^{-1}X \end{aligned}$$

that is, $\rho_m(\{axba(xb)^{-1}\}^{-1}x) = \rho_m(xb\{axba(xb)^{-1}\}^{-1}(axb)^{-1}xb)$. This completes the proof. □

Remark 1. *Since the representation ρ_m comes from the holonomy representation obtained from the ideal triangulation of E given in* [TY]*, the holonomy representation ρ_m of $G(K_n)$ is given by the solution to* (1) *which maximizes the hyperbolic volume of $S^3 \backslash K_n$.*

3. Calculation of the twisted Alexander polynomial

The following is the main result of this paper.

Theorem 1. *The twisted Alexander polynomial of K_n associated to ρ_m is given as follows, where*

$$H = 1 - m^2 s + m^2 s^{2n+1} - s^{2n+2},$$
$$\eta_1 = m\alpha - ms^{2n+1}\alpha + s^{2n}\beta + m^2 s^{2n}\beta,$$
$$\eta_2 = -ms\alpha + ms^{2n+1}\alpha - s^{2n}\beta - s^{2n+1}\beta.$$

(i) *If $n > 2$*

$$\Delta_{K_n,\rho_m}(t) = 1 + \sum_{i=0}^{2n-1} \lambda_i (t^{i+3} + t^{4n-i+3}) + t^{4n+6},$$

where

$$\lambda_i = \begin{cases} \dfrac{s(m^2+1)}{m}\left(s^{i/2} - \dfrac{\eta_1+\eta_2}{H\beta}(s^{i/2+1} - s^{-i/2-1})\right) & \text{if } 0 \le i \le 2n-2 \\ & \text{and } i \text{ is even,} \\ \dfrac{s^{(i-1)/2} - s^{-(i-1)/2}}{s - s^{-1}} & \text{if } 0 \le i \le 2n-2 \\ & \text{and } i \text{ is odd,} \\ \dfrac{s^{n-1} - s^{-(n-1)}}{s-s^{-1}} - \dfrac{(s^2-1)\eta_1}{Hs^n\beta} & \text{if } i = 2n-1. \end{cases}$$

(ii) *If* $n = -1$

$$\Delta_{K_{-1},\rho_m}(t) = \frac{ms(s-1)\alpha + (3m^2+1)\beta}{m^2\beta} - \frac{2(m^2+1)}{m}t + \frac{ms(s-1)\alpha + (3m^2+1)\beta}{m^2\beta}t^2.$$

(iii) *If* $n = -2$

$$\begin{aligned}\Delta_{K_{-2},\rho_m}(t) = 1 + t^6 - \frac{m^2+1}{m}(t+t^5) &+ \left(\frac{ms^2(s^3-1)\alpha + (m^2+1)s\beta}{(m^2-s+m^2s^2)\beta} + \frac{s^2+s+1}{s}\right)(t^2+t^4) \\ &- \frac{2(m^2+1)s((s-1)s^2\alpha + m\beta)}{(m^2-s+m^2s^2)\beta}t^3.\end{aligned}$$

(iv) *If* $n < -2$

$$\Delta_{K_n,\rho_m}(t) = \sum_{i=0}^{-2n-1} \lambda_i(t^i + t^{-4n-2-i}),$$

where

$$\lambda_i = \begin{cases} \dfrac{s^{i/2+1} - s^{-i/2-1}}{s-s^{-1}} & \text{if } i \text{ is even and,} \\ & i \ne 2, -2n-2, \\ \dfrac{1+s+s^2}{s} & \text{if } i = 2, \\ -\dfrac{s^{-n-1}(Hs\beta(s^{2n}-1) + (s^2-1)^2\eta_1)}{H(s-s^{-1})\beta} & \text{if } i = -2n-2, \\ \dfrac{s(m^2+1)}{m}\left(\dfrac{\eta_1+\eta_2}{H\beta}(s^{-(i-1)/2} - s^{(i-1)/2}) - s^{-(i-1)/2-1}\right) & \text{if } i \text{ is odd.} \end{cases}$$

As a result, we can observe following.

COROLLARY 1. *The coefficient of the second highest degree of the twisted Alexander polynomial Δ_{K_n,ρ_1} is real if K_n is a fibered knot i.e. the second coefficients are written by*

$$\frac{s(m^2+1)}{m}\left(s^{0/2}-\frac{\eta_1+\eta_2}{H\beta}(s^{0/2+1}-s^{-0/2-1})\right)=0 \qquad \text{if } n>2$$

$$-\frac{m^2+1}{m}=-2 \qquad \text{if } n=-2$$

$$\frac{s(m^2+1)}{m}\left(\frac{\eta_1+\eta_2}{H\beta}(s^{-(1-1)/2}-s^{(1-1)/2})-s^{-(1-1)/2-1}\right)=-2 \qquad \text{if } n<-2,$$

where $m=1$, which corresponds to their holonomy representations.

Then the above result can be reformulated as follows:

REMARK 2. *Suppose $n \neq 0,1,2$. Then the twisted Alexander polynomial of K_n associated to ρ_m can be rewritten by*

$$\frac{1}{2}\sum_{i=0}^{2|n|-1}\left\{\left(1+\frac{|n|}{n}(-1)^i\right)\kappa_i+\left(1-\frac{|n|}{n}(-1)^i\right)\lambda_i\right\}(t^i+t^{(|n|/n)(4n+1)-1-i})$$
$$+t^{(|n|/2n)(4n+1)-1/2}\left(t^{-2n-3}-\frac{(s^2-1)\eta_1}{Hs^n\beta}(t^{-1}+t)+t^{2n+3}\right),$$

up to $t^{-(3/2n)(n+|n|)}$, where

$$\kappa_i=\frac{s(m^2+1)}{m}\left\{\frac{\eta_1+\eta_2}{H\beta}(s^{-(1/4)(2i+3|n|/n+1)}-s^{(1/4)(2i+3|n|/n+1)})+\frac{|n|}{n}s^{(|n|/4n)(2i+1)-1/4}\right\},$$

$$\lambda_i=\frac{1}{s-s^{-1}}(s^{(1/4)(2i-3|n|/n+1)}-s^{-(1/4)(2i-3|n|/n+1)}).$$

To prove Theorem 1, it suffices to show the following:

PROPOSITION 2. *For simplicity, we put $S=s^n$ and $T=t^n$. The twisted Alexander polynomial $\Delta_{K_n,\rho_m}(t)$ is given by*

$$\frac{S-T^2}{s-t^2}\frac{s}{S}\left(\frac{mst-mStT^2+(1+m^2)(1-s^2)St^2T^2}{m(1-s^2)t^3}+\frac{(1+m^2)(1-sSt^2T^2)(\eta_1+\eta_2)}{Hmt^3\beta}\right)$$

$$+\frac{1-ST^2}{1-st^2}\frac{s}{S}\left(\frac{(1+m^2)(1-s^2)S-mSt+mstT^2}{m(1-s^2)t^3}\right.$$
$$\left.-\frac{(1+m^2)(sS-t^2T^2)(\eta_1+\eta_2)}{Hmt^3\beta}\right)$$
$$+\frac{1}{t^6}+T^4+\frac{(1-s^2)(1+t^2)T^2\eta_1}{HSt^4\beta}.$$

By multiplying $t^{2(|n|-n+1)+(1/2n)(|n|+n)}$ and rearranging this with respect to t, we obtain the formula of Theorem 1, by using

$$\frac{S-T^2}{s-t^2}=\frac{|n|}{n}s^{(1/2)(|n|+n)-1}t^{n-|n|}\sum_{i=0}^{|n|-1}\frac{t^{2i}}{s^i},$$

$$\frac{ST^2-1}{st^2-1}=\frac{|n|}{n}s^{(1/2)(n-|n|)}t^{n-|n|}\sum_{i=0}^{|n|-1}s^it^{2i}.$$

4. Proof of Proposition 2

Recall that

$$\begin{aligned}\pi_1(E_n)&=\langle a,b,x\,|\,\{axba(xb)^{-1}\}^{-1}x=xb\{axba(xb)^{-1}\}^{-1}(axb)^{-1}xb,\\&\qquad x=\{axba(xb)^{-1}\}^n\rangle\\&=\langle a,c\,|\,(acac^{-1})^{n-1}=c(acac^{-1})^{-1}(ac)^{-1}c\rangle.\end{aligned}$$

Then the twisted Alexander polynomial of K_n $(n>2)$ is given by

$$\Delta_{K_n,\rho_m}(t)=\frac{\det\Phi\left(\frac{\partial}{\partial a}(acac^{-1})^{n-1}-\frac{\partial}{\partial a}c(acac^{-1})^{-1}(ac)^{-1}c\right)}{\det\Phi(c-1)},$$

where

$$\begin{aligned}&\Phi\left(\frac{\partial}{\partial a}(acac^{-1})^{n-1}-\frac{\partial}{\partial a}c(acac^{-1})^{-1}(ac)^{-1}c\right)\\&\quad=\sum_{i=1}^{n-1}t^{2(i-1)}\rho_m(\{axba(xb)^{-1}\}^{i-1})\{\rho_m(1)+t^{2(n+1)}\rho_m(axb)\}\\&\qquad+t^{4n+1}\rho_m(xbxba^{-1})+t^{2n-1}\rho_m(xb\{axba(xb)^{-1}\}^{-1})\\&\qquad+t^{-3}\rho_m(xb\{axba(xb)^{-1}\}(axb)^{-1}).\end{aligned}\tag{2}$$

For simplicity, we put

$$\gamma_1 = s\alpha - m\beta, \qquad \gamma_2 = ms\alpha - \beta, \qquad \gamma_3 = m^2 s(sS^2+1)\alpha.$$

By the aid of Mathematica, the first term of the right hand side of (2) is given by

$$\sum_{i=1}^{n-1} t^{2(i-1)}(AXBA(XB)^{-1})^{i-1}(E + t^{2(n+1)}AXB)$$
$$= \begin{pmatrix} \dfrac{(ST^2 - st^2)(St^2\beta T^2 + m\alpha)}{mst^2(st^2-1)\alpha} & -\dfrac{T^2(ST^2 - st^2)(\gamma_1\eta_2 + (m\alpha - \beta)\gamma_3)}{m^2 s(s+1)S(st^2-1)\alpha\beta} \\ \dfrac{mC_1\alpha - St^2T^2C_2\beta}{msS(sS^2+1)t^2(s-t^2)(st^2-1)\alpha} & \dfrac{C_3t^4T^4 + C_4t^2T^4 + C_5t^6T^2 + C_6t^4T^2 + C_7}{(s+1)S^2t^2(s-t^2)(st^2-1)\gamma_3\beta} \end{pmatrix},$$

where

$$C_1 = -t^4 s(s^2-1)S - T^2\{t^2(S^2 - s^4) - s(S^2 - s^2)\},$$

$$C_2 = -t^2(t^2-1)s(s+1)S + T^2\{t^2(S^2 + s^3) + s(S^2 - s)\},$$

$$C_3 = (s^3 + S^2)\gamma_1\eta_2 - \{s^3(ms\alpha + \beta) - S^2(m\alpha - \beta)\}\gamma_3,$$

$$C_4 = -s(s + S^2)\gamma_1\eta_2 + s\{s(ms\alpha + \beta) - S^2(m\alpha - \beta)\}\gamma_3,$$

$$C_5 = -s(s+1)S\{\gamma_1\eta_2 + (\eta_1 + \eta_2 - (1 + m^2S^2 - sS^2)\beta)\gamma_3\},$$

$$C_6 = s(s+1)S\{s\alpha\eta_2 - m(s+1)S^2\beta\gamma_2\},$$

$$C_7 = s(s+1)S(st^2-1)(St^2 - sT^2)\beta\gamma_3.$$

Similarly, the second term of the right hand side of (2) is given by

$$XBXBA^{-1} = \begin{pmatrix} \dfrac{S^2D_1}{\gamma_3\alpha} & \dfrac{msD_1D_2 - (sS^2+1)(sS^2D_1 + m\gamma_3\alpha)\beta^2}{(s+1)\gamma_3\alpha\beta^2} \\ \dfrac{(s+1)D_2}{(sS^2+1)\gamma_3\alpha} & \dfrac{msS^2D_1D_2 + s(sS^2+1)(m^2s\alpha^2 - S^2\beta^2)D_2}{S^2(sS^2+1)\gamma_3\alpha\beta^2} - m \end{pmatrix},$$

where

$$D_1 = -(s+1)\alpha\gamma_2 + m(\eta_1 + \gamma_2 + mS^2\gamma_1)\beta,$$

$$D_2 = -\alpha\eta_2 + mS^2(\eta_1 + mS^2\gamma_1 + \gamma_2)\beta,$$

the third term of the right hand side of (2) is given by

$$XB\{AXBA(XB)^{-1}\}^{-1} = \begin{pmatrix} \dfrac{SE_1}{ms(sS^2+1)\alpha\beta} & -\dfrac{S\gamma_1\gamma_2}{m\alpha\beta} \\ \dfrac{(s+1)E_2}{msS(sS^2+1)^2\alpha\beta} & \dfrac{E_3}{mS(sS^2+1)\alpha\beta} \end{pmatrix},$$

where

$$E_1 = (s^2-1)\alpha\gamma_2 + m(\eta_1 + mS^2\gamma_1 - s\gamma_2)\beta,$$

$$E_2 = (s-1)\alpha\eta_2 + mS^2(\eta_1 + mS^2\gamma_1 - s\gamma_2)\beta,$$

$$E_3 = -s\alpha\eta_2 + m(s+1)S^2\beta\gamma_2,$$

and the fourth term of the right hand side of (2) is given by

$$XB(AXBAXBA(XB)^{-1})^{-1} = \begin{pmatrix} \dfrac{mF_3}{\gamma_3^2\beta^2} & \dfrac{F_4}{m(s+1)\gamma_3\alpha\beta^2} \\ \dfrac{m(s^2-1)F_1F_2}{S^2(sS^2+1)\gamma_3^2\beta^2} & \dfrac{mF_5}{S^2\gamma_3^2\beta^2} \end{pmatrix},$$

where

$$F_1 = m(s+1)S^2(\eta_1 + mS^2\gamma_1)\beta - \eta_2\alpha,$$

$$F_2 = m(s+1)S^2(sS^2+1)\beta^2 - sF_1,$$

$$F_3 = -\{m\beta(\eta_1 + mS^2\gamma_1) + s\gamma_1\gamma_2 - \gamma_2\alpha\}F_2 + ms(s+1)S^2(sS^2+1)\gamma_1\gamma_2\beta^2,$$

$$\begin{aligned} F_4 = {} & (s^2-1)\{m(\eta_1 + mS^2\gamma_1)\beta - \gamma_2\alpha\}F_2 \\ & + \gamma_3\{m\gamma_2\alpha - (m^2\eta_1 + s^2\eta_2 + m^3S^2\gamma_1 - s^2(S^2-1)\gamma_2)\beta - ms\gamma_1\gamma_2\}\alpha, \end{aligned}$$

$$F_5 = (s-1)(sF_1 - m\gamma_3\alpha)F_2 - m^2S^2(sS^2+1)\gamma_3\alpha\beta^2.$$

Therefore, the determinant of the right hand side of (2) is written as

$$\frac{\sum_{i,j} U_{i,j}t^iT^j}{m^3S^2t^6(s-t^2)(st^2-1)\beta^2\iota},$$

where

$$U_{0,0} = U_{4,0} = U_{6,0} = U_{2,4} = U_{10,4} = U_{6,8} = U_{8,8} = U_{12,8} = -m^3sS^2\beta^2\iota,$$

$$U_{2,0} = U_{10,8} = m^3(s^2+1)S^2\beta^2\iota,$$

$$HU_{3,0} \equiv HU_{9,8} \equiv -m^2(m^2+1)sS^2\beta(Hs\beta - (s^2-1)(\eta_1+\eta_2))\iota \bmod r_0,$$

$$U_{5,0} \equiv U_{7,8} \equiv m^2(m^2+1)sS^2\beta^2\iota \bmod r_0,$$

$$
\begin{aligned}
HU_{1,2} &\equiv HU_{11,6} \equiv m^2(m^2+1)(s-1)sS\beta\eta_2\iota \bmod r_0,\\
HU_{2,2} &= HU_{6,2} = HU_{8,2} = HU_{4,6} \equiv HU_{6,6} = HU_{10,6}\\
&\equiv m^3(s^2-1)sS\beta\eta_1\iota \bmod r_1,\\
HU_{3,2} &\equiv HU_{9,6} \equiv m^2(m^2+1)(s-1)S\beta\{HsS^2\beta - s(sS^2+1)\eta_1\\
&\qquad - (s^2S^2+s^2+1)\eta_2\}\iota \bmod r_0,\\
H^2U_{4,2} &\equiv H^2U_{8,6} \equiv m(s-1)sS\{H^2m^3\alpha\beta + H(m^2+1)(m^2s+s+1)\beta\eta_2\\
&\qquad - (m^2+1)^2(s^2-1)\eta_2(\eta_1+\eta_2)\}\iota \bmod r_0,\\
HU_{5,2} &\equiv HU_{7,6} \equiv -m^2(m^2+1)(s-1)sS\beta\eta_2\iota \bmod r_0,\\
HU_{7,2} &\equiv HU_{5,6} \equiv m^2(m^2+1)(s-1)sS\beta(HS^2\beta - (sS^2+1)\eta_1\\
&\qquad - (sS^2-1)\eta_2)\iota \bmod r_1,\\
H^2U_{3,4} &\equiv H^2U_{9,4} \equiv -m^2(m^2+1)(s-1)^2s(s+1)\eta_1\eta_2\iota \bmod r_0,\\
H^2U_{4,4} &= H^2U_{8,4} \equiv m\{H^2m^2(s^2-s+1)S^2\beta^2 + (m^2+1)^2(s-1)^2s\eta_2\\
&\qquad \times (-HS^2\beta + (sS^2+1)\eta_1 + sS^2\eta_2)\}\iota \bmod r_1,\\
H^2U_{5,4} &\equiv H^2U_{7,4} \equiv -(m^2+1)(s-1)s\{(s-1)\eta_2(m^3H\alpha + (m^2+1)\eta_2)\\
&\qquad + m^2S^2H\beta(H\beta - (s+1)(\eta_1+\eta_2))\}\iota \bmod r_0,\\
H^2U_{6,4} &\equiv -2ms(HmS\beta - (m^2+1)(s-1)\eta_2)\\
&\qquad \times (HmS\beta + (m^2+1)(s-1)\eta_2)\iota \bmod r_0,
\end{aligned}
$$

where we put $\iota = m^2s^2(s+1)S(sS^2+1)^3\alpha^3\beta$, and the other $U_{i,j}$'s are 0.

On the other hand, by the aid of Mathematica, we have

$$
\begin{aligned}
\det \Phi(c-1) &= \det\left(t^{2n+1}\rho_m(xb) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\\
&= \frac{mSH\beta + mSHt^2T^4\beta - (m^2+1)(s-1)tT^2\eta_2}{mSH\beta}\\
&\quad - \frac{(S^2-1)tT^2}{mS(sS^2+1)H\alpha\beta} r_1\\
&= \frac{mSH\beta + mSHt^2T^4\beta - (m^2+1)(s-1)tT^2\eta_2}{mSH\beta}.
\end{aligned}
$$

Consequently, we have

$$\Delta_{K_n,\rho_m}(t) = \frac{\sum_{i,j} V_{i,j} t^i T^j}{Hm^2 S t^6 (s-t^2)(st^2-1)\beta}, \tag{3}$$

where

$$\begin{aligned}
&V_{0,0} = V_{4,0} = V_{6,0} = V_{4,4} = V_{6,4} = V_{10,4} = -Hm^2 sS\beta,\\
&V_{2,0} = V_{8,4} = Hm^2(s^2+1)S\beta,\\
&V_{3,0} = V_{7,4} = m(m^2+1)sS\{(s^2-1)(\eta_1+\eta_2) - Hs\beta\},\\
&V_{5,0} = V_{5,4} = Hm(m^2+1)sS\beta,\\
&V_{2,2} = V_{8,2} = m^2 s(s^2-1)\eta_1,\\
&V_{3,2} = V_{7,2} = m(m^2+1)(s-1)s\{(s+1)\eta_1+\eta_2\}\\
&V_{4,2} = V_{6,2} = (s-1)s\{(m^2+1)\eta_2 + Hm^3\alpha\},\\
&V_{5,2} = -2m(m^2+1)(s-1)s\eta_2,
\end{aligned}$$

and the other $V_{i,j}$'s are 0. By the aid of Mathematica, the difference between the right hand side of (3) and the formula in Proposition 2 is equal to

$$\frac{s\zeta_1 + t\zeta_2 - 2t^2\zeta_1 + t^3\zeta_2 + st^4\zeta_1}{Hm^2 S t^3 (s+1)(s-t^2)(st^2-1)\beta} T^2,$$

where

$$\begin{aligned}
&\zeta_1 = m(m^2+1)s(s+1)(HS^2\beta - s(S^2-1)\eta_1 - (sS^2-1)\eta_2),\\
&\zeta_2 = Hm^2 s(m\alpha - ms^2\alpha + s\beta + S^2\beta) - (s^2-1)(m^2\eta_1 + m^2 s^3\eta_1 + s\eta_2 + m^2 s\eta_2).
\end{aligned}$$

Note that $\zeta_1 = 0$ by the definition of H, η_1 and η_2 and that

$$\zeta_2 = m\{(m^2(s^2-s+1)-s)(s^3S^2+1) - Hs(s-1)\}r_0 = 0.$$

This completes the proof of Proposition 2 for $n > 2$.

Remark 3. *For $n < 0$, the knot group $G(K_n)$ is presented by*

$$\begin{aligned}
\pi_1(E_n) &= \langle a, c \mid (acac^{-1})^{n-1} = c(acac^{-1})^{-1}(ac)^{-1}c\rangle\\
&= \langle a, c \mid (acac^{-1})^{-n+1} = c^{-1}(ac)(acac^{-1})c^{-1}\rangle.
\end{aligned}$$

Then, the twisted Alexander polynomial of K_n is given by

$$\Delta_{K_n,\rho_m}(t) = \frac{\det \Phi\left(\frac{\partial}{\partial a}(acac^{-1})^{-n+1} - \frac{\partial}{\partial a}c^{-1}(ac)(acac^{-1})c^{-1}\right)}{\det \Phi(c-1)},$$

where

$$\begin{aligned}\Phi&\left(\frac{\partial}{\partial a}(acac^{-1})^{-n+1} - \frac{\partial}{\partial a}c^{-1}(ac)(acac^{-1})c^{-1}\right)\\ &= \sum_{i=0}^{-n} t^{2i}\rho_m(\{axba(xb)^{-1}\}^i)\{\rho_m(1) + t^{2(n+1)}\rho_m(axb)\} - t^{-2n-1}\rho_m((xb)^{-1})\\ &\quad - t\rho_m((xb)^{-1}axb) - t^{2n+3}\rho_m((xb)^{-1}axbaxb).\end{aligned}$$

By calculating the above formula in the same way as $n > 2$ and multiplying t^{4n-4}, we can obtain the formula of Proposition 2.

Acknowledgement

The author would like to thank professor Yoshiyuki Yokota for supervising and giving helpful comments, professor Teruhiko Soma and professor Manabu Akaho for giving valuable comments. She also would like to thank the referee for useful suggestion.

References

[CF] R. H. Crowell and R. H. Fox, Introduction to knot theory, Springer-Verlag, 1963.

[CGLS] M. Culler, C. M. Gordon, J. Luecke and P. B. Shalen, Dehn Surgery on Knots, Ann. Math., **125** (1987), 237–300.

[DFJ] N. Dunfield, S. Friedl and N. Jackson, Twisted Alexander polynomials of hyperbolic knots, Exp. Math., **21** (2012), 329–352.

[GKM] H. Goda, T. Kitano and T. Morifuji, Twisted Alexander invariants (in Japanese), Sugaku-Memoirs 5 (2006).

[GaMa] S. Garoufalidis and T. W. Mattman, The A-polynomial of the $(-2, 3, 3+2n)$ pretzel knots, New York J. Math., **17** (2011), 269–279.

[GoMo] H. Goda and T. Morifuji, Twisted Alexander polynomial for $SL(2,\mathbb{C})$-representations and fibered knots, C. R. Math. Acad. Sci. Soc. R. Can., **25** (2003), no. 4, 97–101.

[HM] M. Hirasawa and K. Murasugi, Genera and fibredness of Montesinos knots, Pacific J. Math., **225** (2006), no. 1, 53–83.

[KM] T. Kitano and T. Morifuji, Divisibility of twisted Alexander polynomials and fibered knots, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) Vol. IV (2005), 179–186.

[L] X. S. Lin, Representations of knot groups and twisted Alexander polynomials, Acta Math. Sin., **17** (2001), 361–380.

[Ma] Thomas W. Mattman, The Culler-Shalem seminorms of the $(-2, 3, n)$ pretzel knot, J. Knot Theory Ramifications, **11** (2002), 1251.

[Mo1] T. Morifuji, Twisted Alexander polynomials of twist knots for nonabelian representations, Bull. Sci. Math., **132** (2008), no. 5, 439–453.

[Mo2] T. Morifuji, Representations of knot groups into $SL(2, \mathbb{C})$ and twisted Alexander polynomials, Handbook of Group Actions (Vol. I), Advanced Lectures in Mathematics, **31** (2015), 527–576.

[Mu] K. Murasugi, Knot Theory and Its Applications, Birkhauser, 1996.

[O] Y. Ohashi, Fiberedness of pretzel knots and twisted Alexander polynomials (in Japanese), Master thesis (2012).

[T1] A. Tran, Twisted Alexander polynomials of genus one two-bridge knots, preprint 2015, arXiv:1506.05035.

[T2] A. Tran, Adjoint twisted Alexander polynomials of genus one two-bridge knots, J. Knot Theory Ramifications, **25** (2016), 1650065.

[TY] N. Tamura and Y. Yokota, A formula for the A-polynomials of $(-2, 3, 2n+1)$-pretzel knots, Tokyo J. Math., **27** (2004), 263–273.

[W] M. Wada, Twisted Alexander polynomial for finitely presentable groups, Topology, **33** (1994), 241–256.

Airi Aso
Depertment of Mathematical Sciences
Graduate School of Science
Tokyo Metropolitan University
1-1 *Minamiohsawa, Hachioji-shi, Tokyo,* 192-0397 *Japan*
E-mail: aso-airi@ed.tmu.ac.jp