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A note on the products $\alpha_1\beta_2\gamma_t$ and $\beta_1^{r+1}\beta_2\gamma_t$ in the stable homotopy of spheres

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Abstract. In the stable homotopy groups of spheres, we have Greek letter elements due to J. F. Adams [2], L. Smith [12] and H. Toda [13]. Here we study the non-triviality of certain products of the first alpha element, the first and the second beta elements and a gamma element in the homotopy groups.

1. Introduction

Let $\mathscr{S}_{(p)}$ denote the stable homotopy category of spectra localized at a prime number $p > 5$, and $S^0 \in \mathscr{S}_{(p)}$ be the sphere spectrum localized at p. Since S^0 is a generator of $\mathscr{S}_{(p)}$ in a sense, the homotopy groups $\pi_*(S^0)$ play an important role in understanding the category $\mathscr{S}_{(p)}$. The homotopy groups $\pi_*(S^0)$ form a commutative graded algebra with multiplication given by composition. Unfortunately, the structure of $\pi_*(S^0)$ is little known. G. Nishida showed that every element in $\pi_t(S^0)$ for $t > 0$ is nilpotent. We have generators of the groups called Greek letter elements. In this paper, we study whether or not a product of the Greek letter elements $\alpha_1 \in \pi_{q-1}(S^0)$, $\beta_1 \in \pi_{pq-2}(S^0)$, $\beta_2 \in \pi_{(2p+1)q-2}(S^0)$ and $\gamma_t \in \pi_{(tp^2+(t-1)p+t-2)q-3}(S^0)$ for $t \geq 1$ is trivial. Hereafter, we put $q = 2p - 2$ as usual.

In [1], M. Aubry determined the homotopy groups $\pi_*(S^0)$ through total degree less than $(3p^2 + 4p)q$. In particular, we have the following:

Theorem 1.1 ([1]). *$\alpha_1\beta_2\gamma_2$ and $\beta_1^r\beta_2\gamma_2$ for $r < p$ are non-trivial, and $\alpha_1\beta_1\beta_2\gamma_2 = 0$.*

X. Liu showed the following theorems:

Theorem 1.2 ([5]). *The products $\alpha_1\beta_2\gamma_s$ are non-trivial for $2 < s < p$.*

Theorem 1.3 ([14]). *The products $\alpha_1\beta_1\beta_2\gamma_s$ are non-trivial for $2 < s < p$.*

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These two theorems are shown by use of the classical Adams spectral sequence. Thus, the subscript s of γ_s must be greater than two.

Consider the Adams-Novikov spectral sequence $\{E_r^{*,*}(X)\}$ converging to the homotopy groups $\pi_*(X)$ of a spectrum X, and let

$$\bar{\alpha}_1 \in E_2^{1,q}(S^0), \qquad \bar{\beta}_1 \in E_2^{2,pq}(S^0), \qquad \bar{\beta}_2 \in E_2^{2,(2p+1)q}(S^0) \qquad \text{and}$$

$$\bar{\gamma}_t \in E_2^{3,(tp^2+(t-1)p+t-2)q}(S^0) \qquad (t \geq 1)$$

be the elements detecting the Greek letter elements α_1, β_1, β_2 and γ_t, respectively. Observing products of these elements in the E_2-term, we obtained the following theorems:

Theorem 1.4 ([11, Th. 1.1]). *The products* $\alpha_1\beta_1^r\gamma_{up+t} \neq 0$ *if* $1 < t < t+u < p$ *and* $r \leq p-2$.

Theorem 1.5 ([3, Th. 1.4]). *Let* t *be a positive integer with* $p \nmid t(t^2-1)$. *Then,* $\beta_2\gamma_t \neq 0 \in \pi_*(S^0)$.

C.-N. Lee showed that

Theorem 1.6 ([4, Th. 4.1, Th. 4.4]). *Let* $p \geq 7$. *The products* $\beta_1^r\gamma_t$ *and* $\beta_1^{r-1}\beta_2\gamma_t$ *are non-trivial if* $0 < t < p$ *and* $r \leq p-1$. *The product* $\alpha_1\beta_1^r\gamma_t$ *is non-trivial if* $2 \leq t < p$ *and* $r \leq p-2$.

By using the result $\beta_1^{p-2}\beta_2\gamma_2 \neq 0$ of Lee [4], we deduce the non-triviality of the product $\beta_1^{p-2}\beta_2\gamma_{p+2}$:

Theorem 1.7. *Let* t *be an integer with* $1 < t < p$ *or* $t = p+2$. *Then, the products* $\beta_1^r\beta_2\gamma_t$ *are non-trivial for* $0 \leq r \leq p-2$.

Consider the spectra $V(2)_k$ for $k \geq 1$ characterized by the Brown-Peterson homology $BP_*(V(2)_k) = BP_*/(p, v_1, v_2^k)$ (see (2.6)). The spectrum $V(2) = V(2)_1$ is the second Smith-Toda spectrum. It is well known that $\bar{\gamma}_1 = \bar{\alpha}_1\bar{\beta}_{p-1}$, and so $\bar{\alpha}_1\bar{\gamma}_1 = 0$ as well as $\alpha_1\gamma_1 = 0$. If $t = p, p+1$, then $\bar{\gamma}_t = 0 \in E_2^{3,(tp^2+(t-1)p+t-2)q}(V(2))$ (see (3.5), *cf.* [4, Lemma 4.3]).

For products $\bar{\alpha}_1\bar{\beta}_2\bar{\gamma}_t$ in the Adams-Novikov E_2-term for computing $\pi_*(V(2))$, we have

Theorem 1.8. $\bar{\alpha}_1\bar{\beta}_2\bar{\gamma}_t = 0 \in E_2^{6,(tp^2+(t+1)p+t)q}(V(2))$ *for* $t \geq p$.

By use of the May and the Novikov spectral sequences together with Toda's calculation [13] on the May E_1-term, we show the non-triviality of an element $\bar{\alpha}_1\bar{\beta}_2\bar{\gamma}_{p+2} \neq 0 \in E_2^{6,(p^3+3p^2+4p+2)q}(V(2)_3)$ in Lemma 2.20. From this, we extend non-triviality of products of Theorems 1.1 and 1.2 to the following:

THEOREM 1.9. *Let t be an integer with $1 < t < p$ or $t = p+2$. Then,* $\alpha_1\beta_2\gamma_t \neq 0 \in \pi_*(S^0)$.

In the next section, we study the Adams-Novikov E_2-term by use of the May and the Novikov spectral sequences with Toda's calculation [13] on the May E_1-term. We then show the non-triviality of $\alpha_1\beta_2\gamma_{p+2}$ in Theorem 1.9 and the triviality of the products in Theorem 1.8 in Section 3. The last section is devoted to the proof of the non-triviality of the composite $\beta_1^{p-2}\beta_2\gamma_{p+2}$ in Theorem 1.7.

2. The Adams-Novikov E_2-terms

We fix a prime number $p \geq 7$. Let BP denote the Brown-Peterson spectrum at the prime p, and we have a Hopf algebroid

$$(BP_*, BP_*(BP)) = (\mathbb{Z}_{(p)}[v_1, v_2, \dots], BP_*[t_1, t_2, \dots])$$

with structure maps: the left and the right units $\eta_L, \eta_R : BP_* \to BP_*(BP)$, the coproduct $\Delta : BP_*(BP) \to BP_*(BP) \otimes_{BP_*} BP_*(BP)$, the counit $\varepsilon : BP_*(BP) \to BP_*$ and the conjugation $c : BP_*(BP) \to BP_*(BP)$. Here, v_i and t_i are generators of degree $2p^i - 2 = e(i)q$ for $e(i) = \frac{p^i-1}{p-1}$ and $q = 2p-2$. We notice here the following action of the structure maps on the generators:

$$\begin{aligned}
&\eta_R(v_n) \equiv v_n + v_{n-1}t_1^{p^{n-1}} - v_{n-1}^p t_1 \bmod I_{n-1} \qquad (n \geq 2),\\
&\eta_R(v_3) \equiv v_3 + v_2 t_1^{p^2} + v_1 t_2^p - t_1\eta_R(v_2^p) + v_1 w_1(v_2) - v_1^{p^2} t_2 \bmod (p),\\
&\eta_R(v_4) \equiv v_4 + v_3 t_1^{p^3} + v_2 t_2^{p^2} - \eta_R(v_3^p)t_1 - v_2^{p^2} t_2 \bmod I_2,\\
(2.1)\qquad &\Delta(t_n) \equiv \textstyle\sum_{i=0}^n t_i \otimes t_{n-i}^{p^i} + v_{n-1}b_{1,n-2} \bmod I_{n-1} \qquad (n \geq 1),\\
&\Delta(t_4) \equiv \textstyle\sum_{i=0}^4 t_i \otimes t_{4-i}^{p^i} + v_3 b_{1,2} + v_2 b_{2,1} \bmod I_2,\\
&c(t_1) = -t_1, \qquad c(t_2) = t_1^{p+1} - t_2 \qquad \text{and}\\
&\Delta(c(x)) = (c \otimes c)T\Delta(x) \qquad \text{for } x \in BP_*(BP).
\end{aligned}$$

(*cf.* [10, Ch. 4]). Here, $T : BP_*(BP) \otimes BP_*(BP) \to BP_*(BP) \otimes BP_*(BP)$ denotes the switching map given by $T(x \otimes y) = y \otimes x$, I_{n-1} denotes the invariant ideal of BP_* generated by $n-1$ elements $v_0 = p$, $v_1, \dots, v_{n-2}$ $(I_0 = 0)$, $w_1(v_2) = (v_2^p + v_1^p t_1^{p^2} - v_1^{p^2} t_1^p - (v_2 + v_1 t_1^p - v_1^p t_1)^p)/p$, and $b_{1,k}$, $b_{2,k}$ and $b_{3,k} \in BP_*(BP) \otimes_{BP_*} BP_*(BP)$ for $k \geq 0$ are the elements fitting in the following equalities

$$(2.2)\quad \begin{aligned} &d(t_1^{p^{k+1}}) = pb_{1,k}, \quad d(t_2^{p^{k+1}}) = -t_1^{p^{k+1}} \otimes t_1^{p^{k+2}} - v_1^{p^{k+1}} b_{1,0}^{p^{k+1}} + pb_{2,k} \quad \text{and} \\ &d(t_3^{p^{k+1}}) = -t_1^{p^{k+1}} \otimes t_2^{p^{k+2}} - t_2^{p^{k+1}} \otimes t_1^{p^{k+3}} - v_2^{p^{k+1}} b_{1,1}^{p^{k+1}} - v_1^{p^{k+1}} b_{2,0}^{p^{k+1}} + pb_{3,k}, \end{aligned}$$

in which $d(x) = 1 \otimes x + x \otimes 1 - \Delta(x) \in BP_*(BP) \otimes_{BP_*} BP_*(BP)$. By the definition (2.2) and the formulas on $\Delta(t_1)$ and $\Delta(t_2)$ in (2.1), we see that

$$(2.3)\quad \begin{aligned} &d(b_{2,i}) = b_{1,i} \otimes t_1^{p^{i+2}} - t_1^{p^{i+1}} \otimes b_{1,i+1} \quad \text{for } i \geq 0, \qquad \text{and} \\ &d(b_{3,0}) \equiv b_{1,0} \otimes t_2^{p^2} - t_1^p \otimes b_{2,1} + b_{2,0} \otimes t_1^{p^3} - t_2^p \otimes b_{1,2} \bmod (p). \end{aligned}$$

We have the Adams-Novikov spectral sequence:

$$E_2^{s,t}(W) = \mathrm{Ext}_{BP_*(BP)}^{s,t}(BP_*, BP_*(W)) \Rightarrow \pi_{t-s}(W)$$

for a spectrum W. In this paper, we use the cobar complex $\Omega^{*,*}BP_*(W)$ for studying elements of the E_2-term: $E_2^{s,t}(W) = H^{s,t}(BP_*(W))$ (*cf.* [7], [4]). Here,

$$(2.4)\quad H^{s,t}(M) = \mathrm{Ext}_{BP_*(BP)}^{s,t}(BP_*, M)$$

for a $BP_*(BP)$-comodule M. Furthermore, we consider the k-th Smith-Toda spectrum $V(k)$ for $k = 0, 1, 2$ defined by the cofiber sequences

$$(2.5)\quad \begin{aligned} &S^0 \xrightarrow{p} S^0 \xrightarrow{i} V(0) \xrightarrow{j} S^1, \qquad \Sigma^q V(0) \xrightarrow{\alpha} V(0) \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{q+1} V(0) \\ &\text{and} \qquad \Sigma^{(p+1)q} V(1) \xrightarrow{\beta} V(1) \xrightarrow{i_2} V(2) \xrightarrow{j_2} \Sigma^{(p+1)q+1} V(1) \end{aligned}$$

for the maps p, α and β, which induces a multiplication by p, v_1 and v_2 on the BP_*-homologies, respectively ([2], [12], *cf.* [10]). We also consider similar spectra $V(2)_k$ for $k \geq 2$ defined by the cofiber sequences

$$(2.6)\quad \Sigma^{k(p+1)q} V(1) \xrightarrow{\beta^k} V(1) \xrightarrow{\tilde{i}_k} V(2)_k \xrightarrow{\tilde{j}_k} \Sigma^{k(p+1)q+1} V(1).$$

We notice that $V(2)_k$ is a ring spectrum if $k \leq (p-2)/2$ ([9, Lemma 4.1], where it is denoted by L_k). Note that $BP_*(V(k)) = BP_*/I_{k+1}$, and $BP_*(V(2)_k) = BP_*/(p, v_1, v_2^k)$.

Consider a Hopf algebra $\mathscr{T} = \mathbf{Z}/p[t_1, t_2, \dots] = BP_*(BP)/(p, v_1, v_2, \dots)$ with structure maps obtained from $BP_*(BP)$ under the projection $BP_*(BP) \to \mathscr{T}$. May [6] constructed spectral sequences:

$$(2.7)\quad \begin{aligned} &E_1 = H^*(V(L)) \Rightarrow H^*(\mathscr{T}) \qquad \text{and} \\ &E_2 = P(b_{i,j}) \otimes H^*(U(L)) \Rightarrow H^*(V(L)). \end{aligned}$$

Here, L denotes the restricted Lie algebra associated to the Hopf algebra $\mathscr{T}$ and $U(L)$ and $V(L) = U(L)/(\xi(x) - x^p)$ are the enveloping algebras of L

(ξ is the "p operation"). The bidegree of the generator $b_{i,j}$ is $(2, p^{j+1}e(i)q)$, and $b_{i,j}$'s correspond to those given above for $i = 1, 2, 3$. The cohomology $H^*(U(L))$ is isomorphic to the cohomology of the exterior complex $E(t_{i,j} : i \geq 1, j \geq 0)$ over generators $t_{i,j}$ with bidegree $(1, p^j e(i)q)$ along with the differential given by

$$d(t_{i,j}) = \sum_{k=1}^{i-1} t_{i-k,j+k} t_{k,j}. \tag{2.8}$$

In [13], Toda determined $H^{s,t}(U(L))$ for $t - s \leq (p^3 + 3p^2 + 2p + 1)q - 4$, which is additively generated by the unit element 1 and the elements in the table:

h_0	h_1	g_0	k_0	k_0h_0	h_2
1	p	$p+2$	$2p+1$	$2p+2$	p^2
h_2h_0	g_1	l_1	l_2	l_1h_1	k_1
p^2+1	p^2+2p	p^2+2p+3	p^2+3p+1	p^2+3p+3	$2p^2+p$
l_3	k_1h_1	l_1h_2	m_1	m_1h_0	l_4
$2p^2+p+2$	$2p^2+2p$	$2p^2+2p+3$	$2p^2+4p+2$	$2p^2+4p+3$	$3p^2+2p+1$
l_4h_0	l_4h_1	l_4g_0	l_4k_0	$l_4k_0h_0$	h_3
$3p^2+2p+2$	$3p^2+3p+1$	$3p^2+3p+3$	$3p^2+4p+2$	$3p^2+4p+3$	p^3
h_3h_0	h_3h_1	h_3g_0	h_3k_0	$h_3k_0h_0$	g_2
p^3+1	p^3+p	p^3+p+2	p^3+2p+1	p^3+2p+2	p^3+2p^2
g_2h_0	l_5	m_2	m_3	l_6	m_4
p^3+2p^2+1	p^3+2p^2+3p	p^3+2p^2 $+3p+4$	p^3+2p^2 $+4p+1$	p^3+3p^2+p	p^3+3p^2 $+p+2$

Table 2.9

Here, the integer under each element is the degree of it divided by q, and

$$\begin{aligned} &h_i = [t_{1,i}], \qquad g_i = [t_{1,i}t_{2,i}], \qquad k_i = [t_{1,i+1}t_{2,i}], \qquad (i \geq 0); \\ &l_1 = [t_{3,0}t_{2,0}t_{1,0}], \qquad l_2 = [t_{2,1}t_{2,0}t_{1,1}], \qquad l_3 = [t_{3,0}t_{1,2}t_{1,0}], \\ &l_4 = [t_{3,0}t_{2,1}t_{1,2}], \qquad l_5 = [t_{3,1}t_{2,1}t_{1,1}], \qquad l_6 = [t_{2,2}t_{2,1}t_{1,2}]; \end{aligned} \tag{2.10}$$

$$m_1 = [t_{3,0}t_{2,1}t_{2,0}t_{1,1}], \qquad m_2 = [t_{4,0}t_{3,0}t_{2,0}t_{1,0}],$$

$$m_3 = [t_{3,1}t_{2,1}t_{2,0}t_{1,1}], \qquad \text{and} \qquad m_4 = [t_{2,2}t_{3,0}t_{1,2}t_{1,0}].$$

Lemma 2.11. *The cohomology* $H^{5,(p^3+3p^2+3p+1)q}(\mathscr{T})$ *is a subquotient of* $\mathbb{Z}/p\{l_4h_3h_1\}$, *and* $H^{5,(p^3+3p^2+4p+2)q}(\mathscr{T}) = 0$.

Proof. We consider the May spectral sequences (2.7). The module $(E(t_{i,j}))^{5,tq}$ for $t = (p^3 + 3p^2 + ap + a - 2)$ with $a = 3$ or $a = 4$ is generated by the monomials of the form

$$t_{1,0}^{\varepsilon_{1,0}}t_{1,1}^{\varepsilon_{1,1}}t_{1,2}^{\varepsilon_{1,2}}t_{1,3}^{\varepsilon_{1,3}}t_{2,0}^{\varepsilon_{2,0}}t_{2,1}^{\varepsilon_{2,1}}t_{2,2}^{\varepsilon_{2,2}}t_{3,0}^{\varepsilon_{3,0}}t_{3,1}^{\varepsilon_{3,1}}t_{4,0}^{\varepsilon_{4,0}}$$

with $\varepsilon_{i,j} \in \{0,1\}$ satisfying equations

$$5 = \varepsilon_{1,0} + \varepsilon_{1,1} + \varepsilon_{1,2} + \varepsilon_{1,3} + \varepsilon_{2,0} + \varepsilon_{2,1} + \varepsilon_{2,2} + \varepsilon_{3,0} + \varepsilon_{3,1} + \varepsilon_{4,0}, \tag{1}$$

$$1 = \varepsilon_{1,3} + \varepsilon_{2,2} + \varepsilon_{3,1} + \varepsilon_{4,0}, \tag{2}$$

$$3 = \varepsilon_{1,2} + \varepsilon_{2,1} + \varepsilon_{2,2} + \varepsilon_{3,0} + \varepsilon_{3,1} + \varepsilon_{4,0}, \tag{3}$$

$$a = \varepsilon_{1,1} + \varepsilon_{2,0} + \varepsilon_{2,1} + \varepsilon_{3,0} + \varepsilon_{3,1} + \varepsilon_{4,0} \qquad \text{and} \tag{4}$$

$$a - 2 = \varepsilon_{1,0} + \varepsilon_{2,0} + \varepsilon_{3,0} + \varepsilon_{4,0}. \tag{5}$$

These equations implies

$$4 = \varepsilon_{1,0} + \varepsilon_{1,1} + \varepsilon_{1,2} + \varepsilon_{2,0} + \varepsilon_{2,1} + \varepsilon_{3,0} \qquad \text{by (1) and (2),} \tag{6}$$

$$2 = \varepsilon_{1,0} + \varepsilon_{1,1} + \varepsilon_{1,3} + \varepsilon_{2,0} \qquad \text{by (1) and (3),} \tag{7}$$

$$2 = \varepsilon_{1,2} + \varepsilon_{2,1} + \varepsilon_{3,0} - \varepsilon_{1,3} \qquad \text{by (2) and (3), and} \tag{8}$$

$$2 = \varepsilon_{1,1} + \varepsilon_{2,1} + \varepsilon_{3,1} - \varepsilon_{1,0} \qquad \text{by (4) and (5).} \tag{9}$$

The case for $\varepsilon_{3,1} = 0$: In this case, we see that $\varepsilon_{1,1} = \varepsilon_{2,1} = 1$ and $\varepsilon_{1,0} = 0$ by (9). Then,

$$2 = \varepsilon_{1,2} + \varepsilon_{2,0} + \varepsilon_{3,0} \quad \text{by (6)} \qquad \text{and} \qquad \varepsilon_{1,3} + \varepsilon_{2,0} = 1 \quad \text{by (7).}$$

- If $\varepsilon_{1,3} = 1$, then $\varepsilon_{2,0} = 0$, and so $\varepsilon_{1,2} = \varepsilon_{3,0} = 1$, and obtain a monomial $t_{1,1}t_{2,1}t_{1,2}t_{3,0}t_{1,3}$ at degree $(p^3 + 3p^2 + 3p + 1)q$, which yields the element $l_4h_1h_3$.
- If $\varepsilon_{1,3} = 0$, then $\varepsilon_{2,0} = 1$, and so $\varepsilon_{1,2} + \varepsilon_{3,0} = 1$.
  - If $\varepsilon_{1,2} = 1$, then the monomial has a factor $t_{1,1}t_{2,1}t_{2,0}t_{1,2}$ of degree $(2p^2 + 3p + 1)q$, and so we obtain

$$t_{1,1}t_{2,1}t_{2,0}t_{1,2}t_{2,2} \quad \text{at } a=3, \qquad \text{and}$$

$$t_{1,1}t_{2,1}t_{2,0}t_{1,2}t_{4,0} \quad \text{at } a=4.$$

The first monomial gives us the element $l_2g_2 = l_6k_0 \in H^{5,tq}(U(L))$. We name the second monomial x_1.

– If $\varepsilon_{1,2} = 0$, then $\varepsilon_{3,0} = 1$, and the monomial has a factor $t_{1,1}t_{2,1}t_{2,0}t_{3,0}$ of degree $(2p^2+4p+2)q$, and so the monomial is $t_{1,1}t_{2,1}t_{2,0}t_{3,0}t_{2,2}$ at degree $(p^3+3p^2+4p+2)q$. We name it x_2.

The case for $\varepsilon_{3,1} = 1$**:** In this case, $\varepsilon_{1,3} = \varepsilon_{2,2} = \varepsilon_{4,0} = 0$ by (2). By (9), $1 = \varepsilon_{1,1} + \varepsilon_{2,1} - \varepsilon_{1,0}$.

- If $\varepsilon_{1,0} = 1$, then $\varepsilon_{1,1} = \varepsilon_{2,1} = 1$, and the monomial has a factor $t_{1,0}t_{1,1}t_{2,1}t_{3,1}$ of degree (p^3+2p^2+3p+1). Therefore, we have monomials $t_{1,0}t_{1,1}t_{2,1}t_{1,2}t_{3,1}$ at $a=3$ and $t_{1,0}t_{1,1}t_{2,1}t_{3,0}t_{3,1}$ at $a=4$. The first monomial corresponds $l_5h_2h_0$. By Table 2.9, we see that $l_5h_0 = 0$ and the monomial yields nothing. We name the second one x_3.
- If $\varepsilon_{1,0} = 0$, then $1 = \varepsilon_{1,1} + \varepsilon_{2,1}$. This together with (6) implies $3 = \varepsilon_{1,2} + \varepsilon_{2,0} + \varepsilon_{3,0}$, and we obtain $\varepsilon_{1,2} = \varepsilon_{2,0} = \varepsilon_{3,0} = 1$. By (8), $\varepsilon_{2,1} = 0$, and so $\varepsilon_{1,1} = 1$. Therefore, we have $t_{1,1}t_{1,2}t_{2,0}t_{3,0}t_{3,1}$ at degree $(p^3+3p^2+4p+2)q$. We name it x_4.

Now put

$$\tilde{x}_1 = t_{1,1}t_{2,1}t_{2,0}t_{1,2}t_{3,1}t_{1,0} \qquad \text{and} \qquad \tilde{x}_2 = t_{1,1}t_{2,1}t_{2,0}t_{1,2}t_{1,3}t_{3,0}.$$

Then,

$$d(x_1) = \tilde{x}_1 + \tilde{x}_2, \qquad d(x_2) = -\tilde{x}_2, \qquad d(x_3) = -\tilde{x}_1$$
$$\text{and} \qquad d(x_4) = -\tilde{x}_1 + \tilde{x}_2,$$

and

$$d(t_{1,1}t_{2,1}t_{3,0}t_{4,0}) = -x_1 - x_3 - x_2 \qquad \text{and} \qquad d(t_{2,1}t_{2,0}t_{3,0}t_{3,1}) = -x_2 + x_3 - x_4.$$

Thus, the elements x_i for $i = 1,2,3,4$ yield no element of $H^{5,(p^3+3p^2+4p+2)q}(U(L))$. We also have

$$\begin{aligned} &d(t_{1,1}t_{2,1}t_{1,2}t_{4,0} - t_{2,0}t_{2,1}t_{1,2}t_{3,1}) \\ &\quad = -t_{1,1}t_{2,1}t_{1,2}(t_{3,1}t_{1,0} + t_{2,2}t_{2,0} + t_{1,3}t_{3,0}) \\ &\qquad - t_{1,1}t_{1,0}t_{2,1}t_{1,2}t_{3,1} + t_{2,0}t_{2,1}t_{1,2}t_{2,2}t_{1,1} \\ &\quad = -2l_2g_2 + l_4h_3h_1. \end{aligned}$$

$H^{5,tq}(V(L))$ for $t = (p^3 + 3p^2 + ap + a - 2)$ with $a = 3$ or 4 also contains elements obtained from the E_1-term of the May spectral sequence (2.7):

$$b_{1,0}H^{3,t'q}(U(L)) \quad \text{for } t' = t - p = (p^3 + 3p^2 + (a-1)p + a - 2), \qquad \text{and}$$

$$b_{1,0}^2 H^{1,t''q}(U(L)) \quad \text{for } t'' = t - 2p = (p^3 + 3p^2 + (a-2)p + a - 2).$$

The latter module is trivial. We have a monomial of the complex $(E(t_{i,j}))^{3,t'q}$:

$$t_{2,1}t_{3,0}t_{4,0} \qquad (t' = p^3 + 3p^2 + 3p + 2),$$

on which the differential acts by $d(t_{2,1}t_{3,0}t_{4,0}) = t_{2,1}t_{2,0}t_{1,2}t_{4,0} + \cdots \neq 0$, and this monomial yields no element of $H^{3,t'q}(U(L))$. Thus there is no element in these modules.

From Table 2.9, we find no element of the form $xb_{i,j}b_{k,l}$ or $xb_{i,j}$ for $x \in H^*(U(L))$ in our degree. □

In order to study the Adams-Novikov E_2-term, we consider the Novikov spectral sequences

(2.12) $$E_1 = \mathrm{Ext}_{\mathscr{T}}(\mathbf{Z}/p, Q) \Rightarrow E_2^{*,*}(V(0))$$

(*cf.* [1, Lemme in p. 61]) and

(2.13) $$E_1 = \mathbf{Z}/p[v_n] \otimes \mathrm{Ext}_{\mathscr{T}}(\mathbf{Z}/p, Q(n+1)) \Rightarrow \mathrm{Ext}_{\mathscr{T}}(\mathbf{Z}/p, Q(n))$$

(*cf.* [1, (1.4.3)]). Here,

(2.14) $$Q = \mathbf{Z}/p[v_1, v_2, \ldots] \qquad \text{and} \qquad Q(n) = Q/(v_1, \ldots, v_{n-1})$$

are comodules with coactions given by

(2.15) $$\eta(v_n) = \sum_{i=0}^{n} v_i t_{n-i}^{p^i}.$$

We note that

$$\mathrm{Ext}_{\mathscr{T}}(\mathbf{Z}/p, Q(5)) = H^*(\mathscr{T})$$

in our range.

Among the generators (2.10) of $H^*(U(L))$, the elements g_i and k_i for $i \geq 0$, l_2, l_4 and l_6 survive to the Adams-Novikov E_2-term, $E_2^*(V(2)_p)$ by the

Massey products

(2.16)
$$g_i = \langle h_i, h_i, h_{i+1}\rangle, \qquad k_i = \langle h_i, h_{i+1}, h_{i+1}\rangle,$$
$$l_2 = \langle h_0, h_1, g_1\rangle, \qquad l_4 = -2\langle h_2, h_2, h_2, k_0\rangle \qquad \text{and} \qquad l_6 = \langle h_1, h_2, g_2\rangle.$$

These satisfy

(2.17)
$$g_i = \langle h_{i+1}, h_i, h_i\rangle, \qquad 2g_i = -\langle h_i, h_{i+1}, h_i\rangle$$
$$\text{and} \qquad 2k_i = -\langle h_{i+1}, h_i, h_{i+1}\rangle$$

for $i \geq 0$. By a juggling theorem of the Massey products, we also see that

$$h_i g_i = 0, \qquad h_{i+1} g_i = h_i k_i \qquad \text{and} \qquad g_i h_{i+2} = 0.$$

We moreover have elements of the $E_2^{*,*}(V(2)_p)$:

(2.18) $$v_3 h_2 = \langle v_2, h_2, h_2\rangle \qquad \text{and} \qquad x b_{2,0} = \left\langle x, (h_1, h_2), \begin{pmatrix} -b_1 \\ b_0 \end{pmatrix} \right\rangle$$

for an element $x \in E_2^{*,*}(V(2)_p)$ with $xh_1 = 0 = xh_2$. Hereafter, we write b_i for the homology class of $b_{1,i}$ (see also (3.3)). For example, $x = h_1, h_2, g_2$ and $k_1 b_2$. Indeed, $k_1 b_2 h_1 = g_1 h_2 b_2 = g_1 h_3 b_1 = 0$.

Lemma 2.19. *For the spectra $V(2)_k$ in* (2.6), *some of the Adams-Novikov E_2-terms are given as follows:*

$$E_2^{3,(2p^2+p)q}(V(2)_3) = \mathbb{Z}/p\{h_2 b_{2,0}\} \qquad \text{and} \qquad E_2^{2p,(3p^2+p)q}(V(2)_{p-1}) = 0.$$

Proof. For $t \leq 2p^2 + 3p + 2$, $E_2^{*,tq}(V(2)_3)$ is a subquotient of $\mathbb{Z}/p[v_2, v_3] \otimes H^*(\mathscr{T})$ by the spectral sequences (2.12) and (2.13), and $H^*(\mathscr{T})$ is a subquotient of $P(b_{i,j}) \otimes H^*(U(L))$ by the May spectral sequence.

We pick generators with given bidegrees out of the module $\mathbb{Z}/p[v_2, v_3] \otimes P(b_{i,j}) \otimes H^*(U(L))$ as in the following table, where $a, b \in \{0, 1, 2\}$ and $x \in H^{*,*}(U(L))$.

bidegree		$a,\ b$	dim x	x	generators
$(3, (2p^2+p)q)$	$v_2^a v_3^b x$	$a = b = 0$	3	—	—
	$v_2^a v_3^b x b_{i,j}$	$a = b = 0$	1	h_2	$h_2 b_{2,0}$

By (2.18), the element $h_2 b_{2,0}$ yields an element of the Adams-Novikov E_2-term. We easily find only one element k_1 of bidegree $(2, (2p^2+p)q)$ in

$\mathbb{Z}/p[v_2, v_3] \otimes P(b_{i,j}) \otimes H^*(U(L))$. This is an element of $E_2^{2,(2p^2+p)q}(V(2)_3)$, and no differential hit $h_2 b_{2,0}$ in any above spectral sequences. Therefore, $h_2 b_{2,0}$ survives to the E_2-term $E_2^{3,(2p^2+p)q}(V(2)_3)$.

Turn to the second. A monomial of bidegree $(2p, (3p^2+p)q)$ of $\mathbb{Z}/p[v_2, v_3] \otimes P(b_{i,j}) \otimes H^*(U(L))$ has one of the forms $v_2^a v_3^b x b_{2,0}^2 b_{1,0}^{p-2-(1/2)\dim x}$, $v_2^a v_3^b x b_{2,0} b_{1,1} b_{1,0}^{p-2-(1/2)\dim x}$, $v_2^a v_3^b x b_{1,1}^2 b_{1,0}^{p-2-(1/2)\dim x}$, $v_2^a v_3^b x b_{2,0} b_{1,0}^{p-1-(1/2)\dim x}$, $v_2^a v_3^b x b_{1,1} b_{1,0}^{p-1-(1/2)\dim x}$ and $v_2^a v_3^b x b_{1,0}^{p-(1/2)\dim x}$. The degrees of these elements are

monomials	degrees
$v_2^a v_3^b x b_{2,0}^2 b_{1,0}^{p-2-(1/2)\dim x}$	$q\big((p+1)a + (p^2+p+1)b + \deg x + 3p^2 - \frac{p}{2}\dim x\big)$
$v_2^a v_3^b x b_{2,0} b_{1,1} b_{1,0}^{p-2-(1/2)\dim x}$	$q\big((p+1)a + (p^2+p+1)b + \deg x + 3p^2 - p - \frac{p}{2}\dim x\big)$
$v_2^a v_3^b x b_{1,1}^2 b_{1,0}^{p-2-(1/2)\dim x}$	$q\big((p+1)a + (p^2+p+1)b + \deg x + 3p^2 - 2p - \frac{p}{2}\dim x\big)$
$v_2^a v_3^b x b_{2,0} b_{1,0}^{p-1-(1/2)\dim x}$	$q\big((p+1)a + (p^2+p+1)b + \deg x + 2p^2 - \frac{p}{2}\dim x\big)$
$v_2^a v_3^b x b_{1,1} b_{1,0}^{p-1-(1/2)\dim x}$	$q\big((p+1)a + (p^2+p+1)b + \deg x + 2p^2 - p - \frac{p}{2}\dim x\big)$
$v_2^a v_3^b x b_{1,0}^{p-(1/2)\dim x}$	$q\big((p+1)a + (p^2+p+1)b + \deg x + p^2 - \frac{p}{2}\dim x\big)$

Since the degree is $(3p^2+p)q$, we see that $\deg x/q \equiv -a-b \bmod p$, and deduce that $a = b = 0$. Indeed, $\deg x/q \equiv d \bmod p$ with $0 \le d \le 3$, $0 \le a < p-1$ and $0 \le b \le 2$. Thus, $x = g_1$, k_1, and we have a candidate $g_1 b_{2,0} b_{1,0}^{p-2}$ for a generator. Note that $d_{2p-1}(g_1 b_{2,0} b_{1,0}^{p-2}) = g_1 h_2 b_{1,0}^{p-1} = h_1 k_1 b_{1,0}^{p-1}$ in the second May spectral sequence in (2.7). Since $h_1 k_1 \ne 0$ by Table 2.9, we have no generator at the degree. □

Lemma 2.20. *We have a non-zero element* $v_2^2 v_3^p b_0 b_1^2 \in E_2^{6,(p^3+3p^2+4p+2)q}(V(2)_3)$.

Proof. Put $t_0 = p^3 + 3p^2 + 4p + 2$. We consider the element $v_2^2 v_3^p b_0 b_1^2 \in E_2^{6,t_0 q}(V(2)_3)$ by the spectral sequences (2.7), (2.12) and (2.13). For this sake, we compute the Ext group $E = \mathrm{Ext}_{\mathscr{T}}^{5,t_0 q}(\mathbb{Z}/p, Q(2))$ for the comodule $Q(2)$ in (2.14). We study whether or not the element $v_2^2 v_3^p b_0 b_1^2$ is in the image of a differential of the spectral sequences, and so it suffices to consider the modules

$$M(a,b,c) = (v_2^a v_3^b v_4^c H^{5,*}(V(L)))^{5,t_0 q} \subset (P(v_2, v_3, v_4)/(v_2^3) \otimes H^{5,*}(V(L)))^{5,t_0 q}.$$

We read off from Table 2.9 and Lemma 2.11, the module

$$M(a,b,c) \sqsubseteq \begin{cases} \mathbb{Z}/p\{v_4 l_2 b_1\} & (a,b,c)=(0,0,1) \\ \mathbb{Z}/p\{v_3v_4h_2b_0^2, v_3v_4h_1b_0b_1\} & (a,b,c)=(0,1,1) \\ \mathbb{Z}/p\{v_2v_4h_2b_0b_{2,0}, v_2v_4h_1b_1b_{2,0}\} & (a,b,c)=(1,0,1) \\ \mathbb{Z}/p\{v_3l_2b_{2,1}\} & (a,b,c)=(0,1,0) \\ \mathbb{Z}/p\{v_2v_3h_3b_{2,0}^2, v_2v_3h_1b_{2,0}b_{2,1}, v_2v_3k_1h_1b_2, & \\ \qquad v_2v_3h_1b_1b_{3,0}, v_2v_3h_2b_0b_{3,0}\} & (a,b,c)=(1,1,0) \\ \mathbb{Z}/p\{v_3^2h_3b_0b_{2,0}, v_3^2h_1b_2b_{2,0}, v_3^2h_1b_0b_{2,1}\} & (a,b,c)=(0,2,0) \\ \mathbb{Z}/p\{v_2v_3^ph_0b_{2,0}^2\} & (a,b,c)=(1,p,0) \\ \mathbb{Z}/p\{v_2^2v_3^ph_2b_0b_1, v_2^2v_3^ph_1b_1^2\} & (a,b,c)=(2,p,0) \\ \mathbb{Z}/p\{v_3^{p+1}h_0b_0b_{2,0}\} & (a,b,c)=(0,p+1,0) \\ \mathbb{Z}/p\{v_2l_4h_3h_1\} & (a,b,c)=(1,0,0) \\ \mathbb{Z}/p\{v_2^2l_6b_0, v_2^2k_1h_1b_{2,1}, v_2^2h_2b_{3,0}b_{2,0}\} & (a,b,c)=(2,0,0) \\ 0 & \text{otherwise.} \end{cases}$$

Here, we write $A \sqsubseteq B$ if A is a subquotient of B. Let $E(a,b,c)$ denote a submodule of E generated by the elements detected by some elements of $M(a,b,c)$. We first verify which of the elements on the right hand side of the above relation yields an element of $M(a,b,c)$, and then evaluate $E(a,b,c)$ by the spectral sequences (2.13).

We consider the second spectral sequence (2.7). Note that the May filtration of the elements $h_{i,j}$ and $b_{i,j}$ are $2i-1$ and $p(2i-1)$, respectively. Then, the May differential $d_{2p-1}: E_{2p-1}^{s,t,u} \to E_{2p-1}^{s+1,t,u-2p+1}$ of the spectral sequence acts as

$$\begin{aligned} d_{2p-1}(b_{2,i}) &= b_{1,i}h_{i+2} - h_{i+1}b_{1,i+1} \qquad \text{for } i \geq 0, \text{ and} \\ d_{2p-1}(b_{3,0}) &= -h_1b_{2,1} + b_{2,0}h_3 \end{aligned} \tag{2.21}$$

by (2.3).

We start from the modules $M(0,1,1)$, $M(1,0,1)$, $M(1,1,0)$ and $M(2,p,0)$. By (2.21), $h_2b_0^2 = h_1b_0b_1$, $h_2b_0b_{2,0} = h_1b_1b_{2,0}$ and $h_2b_0b_1 = h_1b_1^2$ in $H^*(V(L))$, and

$$\begin{aligned} d_{2p-1}(h_3b_{2,0}^2) &= -2h_3(b_{1,0}h_2 - h_1b_{1,1})b_{2,0} = 2h_3h_1b_{1,1}b_{2,0}, \\ d_{2p-1}(h_1b_{2,0}b_{2,1}) &= -h_1(b_{1,0}h_2 - h_1b_{1,1})b_{2,1} - h_1b_{2,0}(b_{1,1}h_3 - h_2b_{1,2}) \\ &= h_3h_1b_{1,1}b_{2,0}, \\ d_{2p-1}(h_1b_{1,1}b_{3,0}) &= -h_1b_{1,1}(-h_1b_{2,1} + b_{2,0}h_3) = h_3h_1b_{1,1}b_{2,0}, \\ d_{2p-1}(h_2b_{1,0}b_{3,0}) &= -h_2b_{1,0}(-h_1b_{2,1} + b_{2,0}h_3) = 0, \qquad \text{and} \end{aligned}$$

$$\begin{aligned} d_{2p-1}(b_{2,0}b_{3,0}) &= (b_{1,0}h_2 - h_1b_{1,1})b_{3,0} + b_{2,0}(-h_1b_{2,1} + b_{2,0}h_3) \\ &= h_2b_{1,0}b_{3,0} - h_1b_{1,1}b_{3,0} - h_1b_{2,0}b_{2,1} + h_3b_{2,0}^2. \end{aligned}$$

These differentials imply that the rank of the module $M(1,1,0)$ is not greater than three. Therefore, $M(0,1,1) \sqsubseteq \mathbf{Z}/p\{v_3v_4h_2b_0^2\}$, $M(1,0,1) \sqsubseteq \mathbf{Z}/p\{v_2v_4h_2b_{2,0}b_0\}$, $M(1,1,0) \sqsubseteq v_2v_3\mathbf{Z}/p\{h_2b_0b_{3,0}, h_1b_{2,0}b_{2,1} - h_1b_1b_{3,0}, k_1h_1b_2\}$ and $M(2,p,0) \sqsubseteq \mathbf{Z}/p\{v_2^2v_3^ph_2b_0b_1\}$. Furthermore, we have $d_{4p-3}(h_2b_{1,0}b_{3,0}) = -h_2b_{1,0}(b_{1,0}h_{2,2} - h_{2,1}b_{1,2}) = -g_2b_{1,0}^2 + k_1b_{1,0}b_{1,2}$, and $d_{4p-3}(h_1b_{2,0}b_{2,1} - h_1b_{1,1}b_{3,0}) = h_1b_{1,1}(b_{1,0}h_{2,2} - h_{2,1}b_{1,2}) = g_2b_{1,0}^2 - g_1b_{1,1}b_{1,2}$. Therefore, we obtain $M(1,1,0) \sqsubseteq \mathbf{Z}/p\{v_2v_3k_1h_1b_2\}$.

Consider the spectral sequence (2.13). The differentials of the spectral sequences are read off from the structure map (2.15). For example, $d_1(v_4) = v_3h_3$ for $n = 3$ and $d_1(v_3) = v_2h_2$ for $n = 2$. For $M(0,1,1)$, noticing that v_4h_2 is represented by a cocycle $v_4t_1^{p^2} + v_3c(t_2^{p^2}) + v_2t_1^{p^2}t_2^{p^2}$ in the cobar complex $Q(2) \otimes \mathscr{T}$, we compute

$$\begin{aligned} & d(v_4t_1^{p^2} + v_3c(t_2^{p^2}) + v_2t_1^{p^2}t_2^{p^2}) \\ &= \underline{v_3t_1^{p^3} \otimes t_1^{p^2}}_1 + \underline{v_2t_2^{p^2} \otimes t_1^{p^2}}_2 + v_2t_1^{p^2} \otimes c(t_2^{p^2}) - \underline{v_3t_1^{p^3} \otimes t_1^{p^2}}_1 \\ &\quad - v_2t_1^{p^2} \otimes t_2^{p^2} - \underline{v_2t_2^{p^2} \otimes t_1^{p^2}}_2 - v_2t_1^{2p^2} \otimes t_1^{p^3} - v_2t_1^{p^2} \otimes t_1^{p^3+p^2} \\ &= -2v_2t_1^{p^2} \otimes t_2^{p^2} - v_2t_1^{2p^2} \otimes t_1^{p^3}, \end{aligned}$$

in which the underlined terms with a subscript cancel each other out. The cocycle $2t_1^{p^2} \otimes t_2^{p^2} + t_1^{2p^2} \otimes t_1^{p^3}$ appearing in the right hand side of the above computation represents $2g_2 \neq 0 \in \mathrm{Ext}_{\mathscr{T}}(\mathbf{Z}/p, Q(3))$ (see (2.14) for $Q(3)$). It follows that v_4h_2 does not survive to $\mathrm{Ext}_{\mathscr{T}}(\mathbf{Z}/p, Q(2))$ in (2.13). Thus, $E(0,1,1) = 0$.

For $M(1,0,1)$, we compute

$$\begin{aligned} h_3h_2b_{2,0} &= h_3\left\langle h_2, (h_1,h_2), \begin{pmatrix} -b_1 \\ b_0 \end{pmatrix} \right\rangle \\ &= (\langle h_3,h_2,h_1\rangle, \langle h_3,h_2,h_2\rangle)\begin{pmatrix} -b_1 \\ b_0 \end{pmatrix} = g_2b_0 \end{aligned} \tag{2.22}$$

by the juggling theorem in the E_{2p}-term of the second spectral sequence in (2.7) by (2.18) and (2.17). We also note that $\langle h_3,h_2,h_1\rangle = 0$ by considering $d(t_3^p)$. Therefore, $d_1(v_4h_2b_{2,0}b_0) = v_3g_2b_0^2$ in the spectral sequence (2.13) for $n = 3$, and $E(1,0,1) = 0$ follows.

In the spectral sequence (2.13) for $n=2$, we compute

$$d_1(v_3^2g_1b_2)=2v_2v_3h_2g_1b_2=2v_2v_3k_1h_1b_2 \qquad \text{and}$$

$$d_1(v_2v_3^{p+1}b_0b_1)=v_2^2v_3^ph_2b_0b_1,$$

where we use the well known relation $g_1h_2=h_1k_1$. Therefore, the triviality of $E(1,1,0)$ and $E(2,p,0)$ follows.

Since $h_2l_2=0=h_3l_2$ by Table 2.9, we see that

$$l_2b_{2,1}=\left\langle l_2,(h_2,h_3),\begin{pmatrix}-b_2\\ b_1\end{pmatrix}\right\rangle$$

in $H^*(V(L))$ in the same manner as (2.18). Note that $\langle h_2,l_2,h_2\rangle=2l_4h_1$ and $\langle h_2,l_2,h_3\rangle=0$ in $H^*(V(L))$. Therefore, in the spectral sequence (2.13) for $n=2$, we compute $d_1(v_3l_2b_{2,1})=-2v_2l_4h_1b_2\neq 0$ and so $E(0,1,0)=0$.

Since $d_{2p-1}(b_{3,0}b_{1,0})=(-h_1b_{2,1}+b_{2,0}h_3)b_{1,0}$ and

$$d_{2p-1}(h_1b_{2,1}b_{1,0})=-h_1(b_{1,1}h_3-h_2b_{1,2})b_{1,0}=-h_3h_1b_{1,1}b_{1,0},$$

we see that $M(0,2,0)\sqsubseteq \mathbb{Z}/p\{v_3^2h_1b_{2,0}b_2\}$. In the spectral sequence (2.13) for $n=2$,

$$\begin{aligned}d_1(v_3^2h_1b_{2,0})&=2v_2v_3h_2\left\langle h_1,(h_1,h_2),\begin{pmatrix}-b_1\\ b_0\end{pmatrix}\right\rangle\\ &=2v_2v_3\langle h_2,h_1,(h_1,h_2)\rangle\begin{pmatrix}-b_1\\ b_0\end{pmatrix}=2v_2v_3(g_1b_1-2k_1b_0)\end{aligned}$$

by (2.17) and (2.18). It follows that $E(0,2,0)=0$.

In the spectral sequence in (2.7), $d_{2p-1}(k_1b_{3,0})=k_1(-h_1b_{2,1}+b_{2,0}h_3)=-k_1h_1b_{2,1}$ and $k_1h_1b_{2,1}=0\in H^*(V(L))$. By (2.3), we compute the differential $d(t_1^{p^2}\otimes b_{2,0}\otimes b_{3,0})$ in the cobar complex for computing $H^*(V(L))$, and deduce that

$$d_{4p-3}(h_2b_{2,0}b_{3,0})=h_2b_{2,0}(b_{1,0}h_{2,2}-h_{2,1}b_{1,2})=g_2b_{2,0}b_{1,0}-k_1b_{1,2}b_{2,0}$$

in the spectral sequence. Here, $xb_{2,0}$ for $x=g_2$, k_1b_2 are given in (2.18). Thus, $M(2,0,0)\sqsubseteq \mathbb{Z}/p\{v_2^2l_6b_0\}$.

We have $M(1,p,0)=0$ and $M(0,p+1,0)=0$, since

$$d_{2p-1}(h_0b_{2,0})=-h_0(b_{1,0}h_2-h_1b_{1,1})=h_2h_0b_{1,0}.$$

Therefore, $E(1,p,0)=0$ and $E(0,p+1,0)=0$.

Hence, $\mathrm{Ext}_{\mathscr{T}}^{5,t_0q}(\mathbb{Z}/p,Q(2))$ is a subquotient of the module

$$\mathbb{Z}/p\{v_4l_2b_1,v_2l_4h_3h_1,v_2^2l_6b_0\}.$$

We consider the element $v_4 l_2$. By (2.16), $l_2 \in E_2^{*,*}(V(2)_2)$. Let $\bar{l}_2$ denote a cocycle representing l_2 in the cobar complex for computing $E_2^{*,*}(V(2)_2)$. By Table 2.9 together with (2.16), we see that $h_0 l_2 = 0$ and $h_3 l_2 = 0$, and so we have cochains y_i such that $d(y_i) = t_1^{p^i} \otimes \bar{l}_2$ for $i = 0, 3$ in the cobar complex. Then,

$$\begin{aligned} d(v_4\bar{l}_2 - v_3 y_3 + v_3^p y_0) &\equiv v_3 t_1^{p^3} \otimes \bar{l}_2 - v_3^p t_1 \otimes \bar{l}_2 + v_2 t_2^{p^2} \otimes \bar{l}_2 - v_2 t_1^{p^2} \otimes y_3 \\ &\quad - v_3 t_1^{p^3} \otimes \bar{l}_2 + v_3^p t_1 \otimes \bar{l}_2 \\ &\equiv v_2(t_2^{p^2} \otimes \bar{l}_2 - t_1^{p^2} \otimes y_3) \bmod (p, v_1, v_2^3). \end{aligned}$$

Since $t_2^{p^2} \otimes \bar{l}_2 - t_1^{p^2} \otimes y_3$ represents an element of the Massey product $\langle h_2, h_3, l_2 \rangle$, which belongs to $H^{4,(p^3+2p^2+3p+1)q}(U(L))$. Therefore, we deduce that $\langle h_2, h_3, l_2 \rangle = 0$ by Table 2.9, and so we have a cochain z such that $d(z) = t_2^{p^2} \otimes \bar{l}_2 - t_1^{p^2} \otimes y_3$. Now the element $v_4 l_2 b_1$ yields an element of $E_2^{*,*}(V(2)_3)$ represented by $(v_4\bar{l}_2 - v_3 y_3 + v_3^p y_0 - v_2 z) \otimes b_{1,1}$.

The other generators of the module are represented by the Massey products

$$-2v_2\langle h_2, h_2, h_2, k_0 \rangle h_3 h_1 \qquad \text{and} \qquad v_2^2 \langle h_1, h_2, g_2 \rangle b_0$$

in the Adams-Novikov E_2-term $E_2^{*,*}(V(2)_3)$ (*cf.* (2.16)). Therefore, the differentials of (2.12) on these generators act trivially, and $v_2^2 v_3^p b_0 b_1^2$ is not in the image of any differentials of the spectral sequences. □

3. On the product $\alpha_1 \beta_2 \gamma_{p+2}$

We recall the definition of the Greek letter elements. The Greek letter elements in the homotopy groups $\pi_*(S^0)$ are defined by composites

$$\alpha_s = j\alpha^s i, \qquad \beta_s = jj_1\beta^s i_1 i \qquad \text{and} \qquad \gamma_s = jj_1 j_2 \gamma^s i_2 i_1 i \tag{3.1}$$

for the maps in (2.5) and the map $\gamma : \Sigma^{(p^2+p+1)q} V(2) \to V(2)$ inducing a multiplication by v_3 on BP_*-homologies given by Toda [13]. We notice that $(\iota \wedge V(0))\alpha^s i = v_1^s \in BP_*/(p)$, $\beta^s i_1 i = (\iota \wedge V(1)) v_2^s \in BP_*/I_2$ and $(\iota \wedge V(2))\gamma^s i_2 i_1 i = v_3^s \in BP_*/I_3$ for the unit map $\iota : S^0 \to BP$ of the ring spectrum BP. Then by the Geometric Boundary Theorem (*cf.* [10, Th. 2.3.4]), the Greek letter elements (3.1) are detected by those in the Adams-Novikov E_2-term defined by

$$\begin{aligned} &\bar{\alpha}_s = \delta_0(v_1^s) \in E_2^{1,sq}(S^0), \qquad \bar{\beta}_s = \delta_0\delta_1(v_2^s) \in E_2^{2,(sp+s-1)q}(S^0) \qquad \text{and} \\ &\bar{\gamma}_s = \delta_0\delta_1\delta_2(v_3^s) \in E_2^{3,(sp^2+(s-1)p+s-2)q}(S^0). \end{aligned} \tag{3.2}$$

Here $\delta_k : E_2^{*,*}(V(k)) \to E_2^{*+1,*}(V(k-1))$ denotes the connecting homomorphism associated to the cofiber sequences in (2.5) ($V(-1) = S^0$). Traditionally we put

$$(3.3) \qquad h_i = [t_1^{p^i}] \in E_2^{1,p^iq}(S^0) \qquad \text{and} \qquad b_i = [b_{1,i}] \in E_2^{2,p^{i+1}q}(S^0),$$

where $[c]$ denotes the cohomology class of a cocycle $c \in \Omega^{*,*}BP_*$. We note that h_i corresponds to h_i in Table 2.9. Then, by definition, we have well known relations (*cf.* [4], [10]):

$$(3.4) \qquad \bar{\alpha}_1 = h_0, \qquad \bar{\beta}_1 \equiv b_0 \qquad \text{and} \qquad \bar{\beta}_2 \equiv 2v_2b_0 + k_0 \bmod I_2$$

in the E_2-term. Furthermore, it is also showen in [4, Lemma 4.3] that

$$(3.5) \qquad \bar{\gamma}_t = 2\binom{t}{2}v_3^{t-2}h_2b_{2,0} + 3\binom{t}{3}v_3^{t-3}l_4 \bmod I_3 = (p, v_1, v_2)$$

in $E_2^{3,(tp^2+(t-1)p+t-2)q}(S^0) = H^{3,(tp^2+(t-1)p+t-2)q}(BP_*)$, where $h_2b_{2,0}$ and l_4 are given in (2.18) and (2.16). By Lemma 2.19, we have

Lemma 3.6. $\bar{\gamma}_2 = 2h_2b_{2,0} \neq 0 \in E_2^{3,(2p^2+p)q}(V(2)_3)$.

Lemma 3.7. *The element* $\bar{\gamma}_{p+2} \in E_2^{3,(p^3+3p^2+2p)q}(S^0)$ *satisfies that* $\bar{\gamma}_{p+2} \equiv v_3^p\bar{\gamma}_2 \bmod (p, v_1, v_2^3)$.

Proof. The relation $\bar{\gamma}_{p+2} \equiv v_3^p\bar{\gamma}_2$ follows from computation:

$$\delta_2(v_3^{p+2}) \equiv v_3^p\delta_2(v_3^2) \bmod (v_2^6).$$

$$\delta_1\delta_2(v_3^{p+2}) = \delta_1(v_3^p\delta_2(v_3^2) + v_2^5x) \equiv v_3^p\delta_1\delta_2(v_3^2) \bmod (v_1^2, v_2^4).$$

$$\delta_0\delta_1\delta_2(v_3^{p+2}) = \delta_0(v_3^p\delta_1\delta_2(v_3^2) + v_1^2y + v_2^4z) \equiv v_3^p\delta_0\delta_1\delta_2(v_3^2) \bmod (p, v_1, v_2^3),$$

for elements $x \in E_2^{1,*}(V(1))$, and $y, z \in E_2^{2,*}(V(0))$. □

Lemma 3.8. *For the spectrum* $V(2)_3$ *in* (2.6), *we have*

$$h_0k_0\bar{\gamma}_2 = 0 \in E_2^{6,(2p^2+3p+2)q}(V(2)_3).$$

Proof. By the juggling Theorem of the Massey products, (2.18) and Lemma 3.6, we compute

$$h_0k_0\bar{\gamma}_2 = g_0h_1\bar{\gamma}_2 = 2g_0(\langle h_1, h_2, h_1\rangle, \langle h_1, h_2, h_2\rangle)\begin{pmatrix} -b_1 \\ b_0 \end{pmatrix}$$

$$= 4g_0g_1b_1 + 2g_0k_1b_1 = 0$$

in $E_2^{6,(2p^2+3p+2)q}(V(2)_3)$. Indeed, $\langle h_1, h_2, h_1\rangle = -2g_1$ by (2.17), and $g_0g_1 = 0 = g_0k_1$. Therefore, the lemma follows. □

Lemma 3.9. *In the Adams-Novikov E_2-term,*

$$\bar{\alpha}_1\bar{\beta}_2\bar{\gamma}_2 = 4v_2^2 b_0 b_1^2 \in E_2^{6,(2p^2+3p+2)q}(V(2)_3).$$

Proof. By (3.4) and Lemma 3.8, we see that $\bar{\alpha}_1\bar{\beta}_2\bar{\gamma}_2 = 2v_2\bar{\alpha}_1\bar{\beta}_1\bar{\gamma}_2$, which is congruent to $4v_2h_0b_0h_2b_{2,0}$ modulo (p, v_1, v_2^3) by Lemma 3.6. We compute

$$\begin{aligned}\frac{1}{4}v_2\bar{\alpha}_1\bar{\beta}_1\bar{\gamma}_2 &\equiv v_2h_0b_0\left\langle h_2, (h_1, h_2), \begin{pmatrix} -b_1 \\ b_0 \end{pmatrix}\right\rangle \\ &\equiv h_0b_0\langle v_2, h_2, (h_1, h_2)\rangle\begin{pmatrix} -b_1 \\ b_0 \end{pmatrix} \\ &\equiv h_0b_0(\langle v_2, h_2, h_1\rangle, \langle v_2, h_2, h_2\rangle)\begin{pmatrix} -b_1 \\ b_0 \end{pmatrix} \\ &\equiv h_0b_0(-\langle v_2, h_2, h_1\rangle b_1 + \langle v_2, h_2, h_2\rangle b_0) \\ &\equiv -v_2\langle h_2, h_1, h_0\rangle b_0b_1 + \langle v_2, h_2, h_2\rangle h_0b_0b_0 \\ &\equiv v_2^2b_0b_1^2 + v_3h_2h_0b_0^2.\end{aligned}$$

Here, the differential $d(c(t_3))$ (see (2.1)) gives us a relation $\langle h_2, h_1, h_0\rangle \equiv v_2b_1 \bmod I_2$ in the E_2-term. We further see that $h_2h_0b_0^2 = 0 \in H^*(V(L))$, since $d_{2p-1}(h_0b_{1,0}b_{2,0}) = h_0h_2b_{1,0}^2$ in the May spectral sequence. □

Theorem 3.10. $\bar{\alpha}_1\bar{\beta}_2\bar{\gamma}_{p+2} \neq 0 \in E_2^{6,(p^3+3p^2+4p+2)q}(S^0)$.

Proof. By Lemma 3.7, we have $\bar{\gamma}_{p+2} = v_3^p\bar{\gamma}_2 \in E_2^{3,(p^3+3p^2+2p)q}(V(2)_3)$, and so

$$\bar{\alpha}_1\bar{\beta}_2\bar{\gamma}_{p+2} = v_3^p\bar{\alpha}_1\bar{\beta}_2\bar{\gamma}_2 = 4v_2^2v_3^pb_0b_1^2 \in E_2^{6,(p^3+3p^2+4p+2)q}(V(2)_3)$$

by Lemma 3.9. Now the theorem follows from Lemma 2.20. □

Proof of Theorem 1.8. For $t = p$ and $= p+1$, $\bar{\gamma}_t = 0$ by (3.5), and so the proposition holds in these cases. Suppose now $t \geq p+2$. Note that $\bar{\beta}_2 = [\tilde{k}_0] = k_0$ and $\bar{\gamma}_t = 2\binom{t}{2}v_3^{t-2}h_2b_{2,0} + 3\binom{t}{3}v_3^{t-3}l_4$ for $t \geq 2$ in $E_2^*(V(2))$ by (3.4) and (3.5) (*cf.* [4, p. 234], [4, Lemma 4.3]). Here, $BP_*(V(2)) = BP_*/I_3$ and l_4 denotes the generator given in [13, p. 55]. This implies that $\bar{\gamma}_t = v_3^p\bar{\gamma}_{t-p}$ for $t \geq p+2$ in $E_2^*(V(2))$, and we also see $v_3^ph_0 = v_3h_3$ in $E_2^1(V(2))$ by $d(v_4)$, where $h_i \in E_2^{1,p^iq}(V(2))$ is an element represented by a cocycle $t_1^{p^i}$. Therefore, $\bar{\alpha}_1\bar{\beta}_2\bar{\gamma}_t$ is represented by $v_3^{t-p-2}h_3k_0\left(2\binom{t}{2}v_3h_2b_{2,0} + 3\binom{t}{3}l_4\right)$. Here, we see that $h_3k_0h_2b_{2,0} = k_0g_2b_{1,0}$ by (2.22). We also see that $h_3k_0l_4 = h_3h_2m_1$ for the generators in Toda's calculation [13, p. 55]. Since both of k_0g_2 and h_3h_2 are zero by Toda's calculation (see Table 2.9), these imply the triviality of $\bar{\alpha}_1\bar{\beta}_2\bar{\gamma}_t$ for $t \geq p+2$. □

4. Non-triviality of $\beta_1^{p-2}\beta_2\gamma_{p+2}$

We begin with a recollection of some results from [4]: $\Omega^{*,*}BP_*\{a\}$ denotes a quotient complex of the cobar complex $\Omega^{*,*}BP_*$ by a subcomplex generated by monomials $m \otimes t^{E_1} \otimes \cdots \otimes t^{E_n}$ with $\sum_{i=1}^n E_i > (a, 0, \ldots)$. Here, t^E for a sequence $E = (e_1, e_2, \ldots)$ denotes the monomial $t_1^{e_1}t_2^{e_2}\ldots \in BP_*(BP)$, and the set of sequences admits the lexicographical ordering (*cf.* [4, p. 235]).

Then, the gamma elements $\bar{\gamma}_t$ for $t \geq 2$ in the Adams-Novikov E_2-term are represented by a cocycle

$$\tilde{\gamma}_t \equiv -tv_2^{p-3}v_3^{t-1}\tilde{k}_0 \otimes t_1 \bmod J_3 = (p, v_1, v_2^{p-1}) \tag{4.1}$$

in $\Omega^{3,(tp^2+(t-1)p+t-2)q}BP_*\{p^2-1\}$ (*cf.* [4, p. 239]). In this section, we consider a spectrum $V(2)_{p-1}$ in (2.6). Note that $BP_*(V(2)_{p-1}) = BP_*/J_3$.

Theorem 4.2. *$\bar{\beta}_1^{p-2}\bar{\beta}_2\bar{\gamma}_{p+2} \neq 0 \in E_2^{2p+1,tq}(S^0)$ for $t = p^3+4p^2+2p+1$.*

Proof. Let $G \in C = \Omega^{2p+1,tq}BP_*$ be a cocycle representing the element $\bar{\beta}_1^{p-2}\bar{\beta}_2\bar{\gamma}_{p+2}$. Then, $G \equiv v_3^pG_2 \bmod J_3$ for a cochain

$$G_2 = -2v_2^{p-3}v_3\tilde{k}_0 \otimes t_1 \otimes (2v_2b_{1,0} + \tilde{k}_0) \otimes b_{1,0}^{\otimes(p-2)}$$

in $\bar{C} = \Omega^{2p+1,(3p^2+p+1)q}BP_*\{p^2+2\}$ by (3.4) and (4.1). Note that G_2 is the cochain $\mathscr{D}$ of [4, p. 240] for $t = 2$, which is shown not to be a coboundary in $\bar{C}/J_3$. We claim that

(4.3) $\quad$ G has no term with v_4 as a factor modulo J_3.

Indeed, if $G = v_3^pG_2 + v_4w + w' \bmod J_3$ for $w, w' \in \Omega^*BP_*/(J_3+(v_4))$, then, applying the differential d to the equality, we obtain $0 = v_3^pd(G_2) + d(v_4) \otimes w + v_4d(w) + d(w')$. Since $d(G_2)$, $d(v_4)$ and $d(w')$ have no term with v_4, we deduce that $d(w) = 0$. Therefore, $[w] \in E_2^{2p,(3p^2+p)q}(V(2)_{p-1})$, which is zero by Lemma 2.19. It follows that there is a cochain $\bar{w}$ such that $w = d(\bar{w})$. So replace v_4w by $d(v_4) \otimes \bar{w}$ so that G has no term with factor v_4 modulo J_3.

Suppose that there is a cocycle $y \in \Omega^{2p,tq}BP_*$ such that $d(y) = G$ in C. Put $y = y_1 + v_4y_2 + v_3^py_3 + z$ for $y_i = \sum_{a,b} v_2^av_3^by_{i,a,b}$ $(i = 1, 2, 3)$ with $y_{i,a,b} \in \Omega^{2p,*}BP_*/I_5$ and $z \in J_3\Omega^{2p,*}BP_*$. By a similar argument showing (4.3), we replace v_4y_2 by a linear combination of terms without factor v_4. Thus we may put $y = y_1 + v_3^py_3 + z$. By (2.1), we see that $d(t_i) \in \Omega^2BP_*/J_3\{p^2+2\}$ has the only one term $v_2b_{1,1}$ if $i = 3$, and $v_2b_{2,1}$ if $i = 4$ with factors v_2 and v_3. It follows that for $x \in \Omega^{2p,uq}BP_*/I_5$ with $u \leq t$, $d(x) \in (\mathbb{Z}/p)\{1, v_2\} \otimes \Omega^{2p+1,uq}BP_*/I_5\{p^2+2\}$ by degree reason. Indeed, $v_2^2b_{1,1}^2 = 0 \in \Omega^{4,2e(3)q}BP_*/I_2\{p^2+2\}$ and $v_2^2b_{2,1}^2$ has an internal degree greater than tq. Since $d(v_3^b) =$

$bv_2v_3^{b-1}t_1^{p^2}$ in $\Omega^{1,*}BP_*/J_3\{p^2+2\}$ by (2.1), we see that

$$d(y) = d(y_1) + v_3^p d(y_3) = v_3^p G_2 \in \Omega^{2p+1,tq}BP_*/J_3\{p^2+2\}.$$

Here, we notice that $d(z) \equiv 0 \bmod J_3$, since J_3 is an invariant ideal. From the equality, we see that $d(y_1) = 0$ and $d(y_3) = G_2$ in $\Omega^{2p+1,(3p^2+p+1)q}BP_*/J_3\{p^2+2\}$. Thus, $G_2 (= \mathscr{D}$ in [4, p. 240]) is a coboundary in the complex. This contradicts to the conclusion of the proof of [4, Th. 4.1]. □

COROLLARY 4.4. $\beta_1^{p-2}\beta_2\gamma_{p+2} \neq 0 \in \pi_{(p^3+4p^2+2p)q-3}(S^0)$.

PROOF. By virtue of Theorem 4.2, it suffices to show that there is no element $x \in E_2^{2,(t-1)q}(S^0)$ such that $d_{2p-1}(x) = \bar{\beta}_1^{p-2}\bar{\beta}_2\bar{\gamma}_{p+2}$ in the Adams-Novikov spectral sequence. In [7, Th. 2.6], it is shown that the E_2-term $E_2^{2,*}(S^0)$ is generated by the elements $\bar{\beta}_{sp^i/j,k+1}$ for integers $p \nmid s \geq 1$, $i,k \geq 0$, $j \geq 1$, subject to $j \leq p^i$ if $s = 1$, $p^k \mid j \leq a_{i-k}$ and $a_{i-k-1} < j$ if $p^{k+1} \mid j$, where $a_0 = 1$, $a_n = p^n + p^{n-1} - 1$ for $n \geq 1$. The internal degree of the element $\bar{\beta}_{sp^i/j,k+1}$ is $(sp^i(p+1) - j)q$, and we have an equation $t - 1 = sp^i(p+1) - j$ to find the element x. Note that $sp^i - j \geq 0$, and we have $2p^3 > sp^{i+1}$ and so $i \leq 2$. Thus, we obtain the only solution $(i,j,s) = (1,p,p+3)$ of the equation. In this case, $k = 0$ by the relation $p^k \mid j \leq a_{i-k}$. The element $\bar{\beta}_{(p+3)p/p} (= \bar{\beta}_{(p+3)p/p,1})$ is a permanent cycle by [8]. Thus, we have no such element x, and hence $\bar{\beta}_1^{p-2}\bar{\beta}_2\bar{\gamma}_{p+2}$ is not in the image of the differential d_{2p-1} of the spectral sequence. □

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