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A note on the topology of arrangements for a smooth plane quartic and its bitangent lines

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ABSTRACT. In this paper, we give a Zariski triple of the arrangements for a smooth quartic and its four bitangents. A key criterion to distinguish the topology of such curves is given by a matrix related to the height pairing of rational points arising from three bitangent lines.

1. Introduction

Let $(\mathscr{B}^1, \mathscr{B}^2)$ be a pair of reduced plane curves. The pair $(\mathscr{B}^1, \mathscr{B}^2)$ is said to be a *Zariski pair* if it satisfies the following two conditions:

(i) For each i, there exists a tubular neighborhood $T(\mathscr{B}^i)$ of $\mathscr{B}^i$ such that $(T(\mathscr{B}^1), \mathscr{B}^1)$ is homeomorphic to $(T(\mathscr{B}^2), \mathscr{B}^2)$.

(ii) There exists no homeomorphism between $(\mathbb{P}^2, \mathscr{B}^1)$ and $(\mathbb{P}^2, \mathscr{B}^2)$.

An N-ple $(\mathscr{B}_1, \dots, \mathscr{B}_N)$ is called a Zariski N-ple if $(\mathscr{B}_i, \mathscr{B}_j)$ is a Zariski pair for any $1 \le i < j \le N$. The first condition for a Zariski pair can be replaced by *the combinatorics* (or the combinatorial type) of $\mathscr{B}^i$. For the precise definition of the combinatorics, see [2] (It can also be found in [17]). Since the combinatorics is more tractable, we always consider the combinatorics rather than the homeomorphism type of $\mathscr{B}^i$. In [18], Zariski first finds that the topology of a pair $(\mathbb{P}^2, \mathscr{B})$ is not determined by the combinatorics of $\mathscr{B}$ in the case where $\mathscr{B}$ is an irreducible sextic with 6 cusps as its singularities. We refer to [2] for results on Zariski pairs before 2008. Within these several years, new approaches to study Zariski pairs for reducible plane curves have been introduced, such as (a) linking sets ([9]), (b) splitting types ([3]), (c) splitting and connected numbers ([15, 16]) and (d) the set of subarrangements of $\mathscr{B}$ ([4, 5]).

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In [9, 4], Zariski pairs for a smooth cubic and its k-inflectional tangents ($k \geq 4$) are investigated based on the method (d) as above. This generalizes E. Artal's Zariski pair for a smooth cubic and its three inflectional tangents given in [1]. In [16], Shirane introduces *connected numbers* and generalizes E. Artal's example to smooth curves of higher degree.

Also, in constructing plane curves which can be candidates for Zariski pairs, the first and the second authors introduce a new method by using the geometry of sections and multi-sections of an elliptic surface ([5, 6, 17]). In [3, 5, 6], with the methods (b) and (d), they give some examples for Zariski N-plet for arrangements of curves with low degrees.

In this article, we consider Zariski pairs for a smooth quartic and its bitangents, which can be considered not only as a continuation of previous studies (e.g., [4]), but also as a new point of view for such a classically well-studied object.

A smooth quartic $\mathcal{Q}$ and its 28 bitangents have been studied intensively by various authors and there are a lot of results on them. A detailed account of the history of the study of quartic curves and their bitangents can be found in [8, Chapter 6]. As for Zariski pairs, however, there do not seem to be any results except a Zariski pair for a smooth quartic and its three bitangents by E. Artal and J. Vallès, about which the authors were informed via private communication. In this article, we study such objects through the Mordell-Weil lattices, the connected numbers and the set of subarrangements. Here are our main results:

THEOREM 1.1. *Consider the following two combinatorial types of arrangements consisting of a smooth quartic $\mathcal{Q}$ and some of its bitangents as follows:*

(a) *the quartic $\mathcal{Q}$ and three of its bitangent lines which are non-concurrent,*

(b) *the quartic $\mathcal{Q}$ and four of its bitangent lines, none of three of which are concurrent.*

Then the following statements hold:

(i) *there exists a Zariski pair for the arrangement* (a)*,*

(ii) *there exists a Zariski triple for the arrangement* (b).

The first statement has already been claimed by E. Artal and J. Vallès. Yet we believe that our proof based on the theory of Mordell-Weil lattices is different from that of theirs and is new. Hence we believe that it is worthwhile to present it here.

In order to explain how we prove Theorem 1.1, we need some preparation. Let $\mathcal{Q}$ be a smooth quartic and choose a point z_o of $\mathcal{Q}$. We can associate a rational elliptic surface $S_{\mathcal{Q},z_o}$ (see [5, 2.2.2], [17, Section 4]) to $\mathcal{Q}$ and z_o, which is given as follows:

(i) Let $f_{\mathscr{Q}} : S_{\mathscr{Q}} \to \mathbb{P}^2$ be the double cover branched along $\mathscr{Q}$. Since $\mathscr{Q}$ is smooth, $S_{\mathscr{Q}}$ is smooth.

(ii) The pencil of lines passing through z_o on $\mathbb{P}^2$ gives rise to a pencil Λ_{z_o} of curves of genus 1 with a unique base point $(f_{\mathscr{Q}})^{-1}(z_o)$.

(iii) Let $\nu_{z_o} : S_{\mathscr{Q}, z_o} \to S_{\mathscr{Q}}$ be the resolution of the indeterminancy for the rational map induced by Λ_{z_o}. We denote the induced morphism $\varphi_{\mathscr{Q}, z_o} : S_{\mathscr{Q}, z_o} \to \mathbb{P}^1$, which gives a minimal elliptic fibration whose generic fiber is denoted by $E_{\mathscr{Q}, z_o}$. Note that $E_{\mathscr{Q}, z_o}$ is an elliptic curve over $\mathbb{C}(\mathbb{P}^1) \cong \mathbb{C}(t)$. The map ν_{z_o} is a composition of two blowing-ups and the exceptional curve for the second blowing-up gives rise to a section O of $\varphi_{\mathscr{Q}, z_o}$. Note that we have the following diagram:

$$\begin{array}{ccc} S_{\mathscr{Q}} & \xleftarrow{\nu_{z_o}} & S_{\mathscr{Q}, z_o} \\ \Big\downarrow{\scriptstyle f_{\mathscr{Q}}} & & \Big\downarrow{\scriptstyle f_{\mathscr{Q}, z_o}} \\ \mathbb{P}^2 & \xleftarrow[q_{z_o}]{} & (\mathbb{P}^2)_{z_o}, \end{array}$$

where $f_{\mathscr{Q}, z_o}$ is a double cover induced by the quotient under the involution $[-1]_{\varphi_{\mathscr{Q}, z_o}}$ on $S_{\mathscr{Q}, z_o}$, which is given by the inversion with respect to the group law on the generic fiber. The morphism q_{z_o} is a composition of two blowing-ups over z_o.

In what follows,

we choose z_o so that the tangent line l_{z_o} at z_o is neither a bitangent line nor a line with intersection multiplicity 4.

Under this situation, we claim that any bitangent line L of $\mathscr{Q}$ gives rise to two sections $s_L^{\pm}$. On the generic fiber $E_{S_{\mathscr{Q}, z_o}}$, we obtain two $\mathbb{C}(t)$-rational points $\pm P_L$ by restricting these sections to $E_{S_{\mathscr{Q}, z_o}}$.

Let us explain how to prove Theorem 1.1 (i). Let L_i $(i = 1, 2, 3)$ be three distinct bitangent lines to $\mathscr{Q}$ and let $\pm P_i$ be the rational points obtained from L_i respectively. Put $\triangle = L_1 + L_2 + L_3$. We then consider the connected number $c_{f_{\mathscr{Q}}}(\triangle)$ ([16] or see §1) in order to distinguish the topology of $\mathscr{Q} + \triangle$. In this article, we give a criterion for $c_{f_{\mathscr{Q}}}(\triangle)$ to be 1 or 2 by using a matrix related to the height pairing $\langle P_i, P_j \rangle$ defined by Shioda ([13]).

As for Theorem 1.1 (ii), we consider all subarrangements of type $\mathscr{Q} + \triangle$ to distinguish the topology of $\mathscr{Q}$ and its four bitangent lines.

The organization of this note is as follows. In §1, we give a brief summary on tools and methods to prove Theorem 1.1. We give a key criterion in §2. Our proof of Theorem 1.1 is given in §3 where we give an explicit example in the case when $\mathscr{Q}$ is the Klein quartic.

2. Preliminaries

In this section, we introduce various notions which we will use to prove Theorem 1.1. The first is the *connected number* introduced by T. Shirane in [16], which will be the key tool in distinguishing the Zariski pair that is claimed to exist in Theorem 1.1 (i). Another is the method considered and refined in [4], where the analysis of subarrangements effectively distinguishes arrangements with many irreducible components. This method distinguishes the Zariski triple that is claimed to exist in Theorem 1.1 (ii). Finally, we introduce the theory of Mordell-Weil lattices, which enables us to conduct the computations needed to apply the above two.

2.1. Connected Numbers. In [16], the *connected number* is defined for a wide class of varieties, but in this subsection we restate the definition and propositions to fit our setting for the sake of simplicity. The following are simplified versions of [16, Definition 2.1, Proposition 2.3].

Definition 2.1. *Let $\phi : X \to \mathbb{P}^2$ be a double cover of the projective plane with smooth branch locus $B \subset \mathbb{P}^2$. Let $C \subset \mathbb{P}^2$ be a plane curve whose irreducible components are not contained in B and assume that $C \setminus B$ is connected. Under this setting, the number of connected components of $\phi^{-1}(C \setminus B)$ is called the connected number of C with respect to ϕ, and will be denoted by $c_\phi(C)$.*

Note that we will often omit "with respect to ϕ" when it is apparent from the context. Also, note that since we are considering double covers only, $c_\phi(C) = 1$ or 2. The key proposition of connected numbers that will be used in distinguishing the topology of plane curves is the following:

Proposition 2.1. *For each $i = 1, 2$, let $\phi_i : X_i \to \mathbb{P}^2$ be a double cover of $\mathbb{P}^2$ with smooth branch locus $B_i \subset \mathbb{P}^2$ and let C_i be a plane curve whose irreducible components are not contained in B_i, such that $C_i \setminus B_i$ is connected. If there exists a homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ with $h(B_1) = B_2$ and $h(C_1) = C_2$ then $c_{\phi_1}(C_1) = c_{\phi_2}(C_2)$.*

Proof. As we are considering double covers only, the assumptions of Proposition 2.3 in [16] are necessarily satisfied if a homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ with $h(B_1) = B_2$ exists. Hence, our statement follows. $\square$

2.2. Distinguishing the embedded topology of plane curves through subarrangements. In [4], we formulated a method to study the topology of reducible plane curves via subarrangements. We here explain its simplified version which fits our case. Let $\mathcal{Q}$ be a smooth quartic and L_i $(i = 1, \dots, 28)$ be its bitangents. Choose a subset $I \subseteq \{1, \dots, 28\}$ and put $\mathcal{L}_I := \sum_{i \in I} L_i$.

Define

$$\underline{\mathrm{Sub}}_{\triangle}(\mathscr{Q}, \mathscr{L}_I) := \left\{ \mathscr{Q} + \sum_{k=1}^{3} L_{i_k} \,\middle|\, \forall \{i_1, i_2, i_3\} \subset I \right\}.$$

Define a map $c_I : \underline{\mathrm{Sub}}_{\triangle}(\mathscr{Q}, \mathscr{L}_I) \to \{1,2\}$:

$$c_I : \underline{\mathrm{Sub}}_{\triangle}(\mathscr{Q}, \mathscr{L}_I) \ni \mathscr{Q} + \sum_{k=1}^{3} L_{i_k} \mapsto c_{f_{\mathscr{Q}}}\left(\sum_{k=1}^{3} L_{i_k} \right) \in \{1,2\}$$

where $f_{\mathscr{Q}}$ is the double cover of $\mathbb{P}^2$ branched along $\mathscr{Q}$. Chose two subset $I_1, I_2 \subseteq \{1, \ldots, 28\}$ and let $\mathscr{B}_i := \mathscr{Q} + \mathscr{L}_{I_i}$. If there exists a homeomorphism $h : (\mathbb{P}^2, \mathscr{B}_1) \to (\mathbb{P}^2, \mathscr{B}_2)$, as $h(\mathscr{Q}) = \mathscr{Q}$ and $h(\mathscr{L}_1) = \mathscr{L}_2$ necessarily hold, it induces a map $h_\natural : \underline{\mathrm{Sub}}_{\triangle}(\mathscr{Q}, \mathscr{L}_{I_1}) \to \underline{\mathrm{Sub}}_{\triangle}(\mathscr{Q}, \mathscr{L}_{I_2})$ such that $c_{I_2} = c_{I_1} \circ h_\natural$:

$$\begin{tikzcd}
\underline{\mathrm{Sub}}_{\triangle}(\mathscr{Q}, \mathscr{L}_{I_1}) \arrow[d, "h_\natural"'] \arrow[dr, "c_{I_1}"] & \\
\underline{\mathrm{Sub}}_{\triangle}(\mathscr{Q}, \mathscr{L}_{I_2}) \arrow[r, "c_{I_2}"'] & \{1,2\}
\end{tikzcd}$$

Hence, as in [4, Proposition 1.2], we have the following proposition:

PROPOSITION 2.2. *With the same notation as above, if $\mathscr{B}_1$ and $\mathscr{B}_2$ have the same combinatorics and $\#c_{I_1}^{-1}(1) \neq \#c_{I_2}^{-1}(1)$, then $(\mathscr{B}_1, \mathscr{B}_2)$ is a Zariski pair.*

2.3. Elliptic surfaces and Mordell-Weil lattices. As for basic references about elliptic surfaces and Mordell-Weil lattices, we refer to [10, 11, 13]. In particular, for those on rational elliptic surfaces, we refer to [12]. In this article, by an *elliptic surface*, we always mean the same notion as in [13, 5]. Namely it means a smooth projective surface S with a relatively minimal genus 1 fibration $\varphi : S \to C$ over a smooth projective curve C with a section $O : C \to S$, which we identify with its image, and at least one singular fiber. Let $\mathrm{Sing}(\varphi) = \{v \in C \mid \varphi^{-1}(v) \text{ is singular}\}$. For $v \in \mathrm{Sing}(\varphi)$, we put $F_v = \varphi^{-1}(v)$. We denote its irreducible decomposition by $F_v = \Theta_{v,0} + \sum_{i=1}^{m_v - 1} a_{v,i} \Theta_{v,i}$, where m_v is the number of irreducible components of F_v and $\Theta_{v,0}$ is the unique irreducible component with $\Theta_{v,0} \cdot O = 1$. We call $\Theta_{v,0}$ the *identity component*. The classification of singular fibers is well-known ([10]). We use the Kodaira notation for the types of singular fibers. Let $\mathrm{MW}(S)$ be the set of sections of $\varphi : S \to C$. We have $\mathrm{MW}(S) \neq \varnothing$ as $O \in \mathrm{MW}(S)$. By [10, Theorem 9.1], $\mathrm{MW}(S)$ is an abelian group with O acting as the zero element. We call $\mathrm{MW}(S)$ the Mordell-Weil group. On the other hand, the generic fiber E_S of $\varphi : S \to C$ is a curve of genus 1 over $\mathbb{C}(C)$, the rational function field of C. The restriction of O to E gives rise to a $\mathbb{C}(C)$-rational point of E, and one can regard E

as an elliptic curve over $\mathbb{C}(C)$, the restriction of O being the zero element. The group $\mathrm{MW}(S)$ can be identified with the group of $\mathbb{C}(C)$-rational points $E(\mathbb{C}(C))$ canonically. For $s \in \mathrm{MW}(S)$, we denote the corresponding rational point by P_s. Conversely, for an element $P \in E(\mathbb{C}(C))$, we denote the corresponding section by s_P.

In [13], a lattice structure on $E(\mathbb{C}(C))/E(\mathbb{C}(C))_{\mathrm{tor}}$ is defined by using the intersection pairing on S through $P \mapsto s_P$. In particular, $\langle\, , \rangle$ denotes the height pairing and Contr_v denotes the contribution term given in [13] in order to compute $\langle\, , \rangle$.

For the elliptic surface $\varphi_{\mathcal{Q},z_o} : S_{\mathcal{Q},z_o} \to \mathbb{P}^1$ in the Introduction, $\varphi_{\mathcal{Q},z_o}$ has a unique reducible singular fiber F_∞, whose type is either I_2 or III and all other singular fibers are irreducible. Let $F_\infty = \Theta_{\infty,0} + \Theta_{\infty,1}$ be the irreducible decomposition. Then for $P_1, P_2 \in E_{\mathcal{Q},z_o}(\mathbb{C}(t))$, we have

$$\langle P_1, P_2 \rangle := 1 + s_{P_1} \cdot O + s_{P_2} \cdot O - s_{P_1} \cdot s_{P_2} - \begin{cases} \frac{1}{2} & \text{if } \Theta_{\infty,1} \cdot s_{P_1} = \Theta_{\infty,1} \cdot s_{P_2} = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Here, the symbol '$\cdot$' denotes the intersection product of divisors.

3. The height pairing and intersection number of sections

3.1. Connected numbers of three bitangents. Let $\mathcal{Q}$ be a smooth plane quartic. We choose homogeneous coordinates $[T, X, Z]$ of $\mathbb{P}^2$ such that $z_o = [0, 1, 0]$ and $Z = 0$ is the tangent line of $\mathcal{Q}$ at z_o. Then we may assume that $\mathcal{Q}$ is given by a homogeneous polynomial $F_{\mathcal{Q}}(T, X, Z)$ of the form

$$F_{\mathcal{Q}}(T, X, Z) = ZX^3 + p(T, Z)X^2 + q(T, Z)X + r(T, Z).$$

Then the affine part of $\mathcal{Q}$, i.e., the part with $Z \neq 0$ is given by

$$F_{\mathcal{Q}}(t, x, 1) = x^3 + p(t, 1)x^2 + q(t, 1)x + r(t, 1).$$

Then let $\varphi_{\mathcal{Q},z_o} : S_{\mathcal{Q},z_o} \to \mathbb{P}^1$ be the rational elliptic surface as in the Introduction and let $E_{\mathcal{Q},z_o}$ be the generic fiber of $\varphi_{\mathcal{Q},z_o}$. Then, by [13, Theorem 10.4], we have

$$E_{\mathcal{Q},z_o}(\mathbb{C}(t)) \cong E_7^*,$$

where E_7^* is the dual lattice of the root lattice E_7. By [14], $E_{\mathcal{Q},z_o}(\mathbb{C}(t))$ contains 56 $\mathbb{C}(t)$-rational points $P = (x, y)$ of the form:

$$x = at + b, \qquad y = ct^2 + dt + e.$$

Since $-P = (x, -y)$, we denote them by

$$\pm P_i = (x_i, \pm y_i) = (a_i t + b_i, \pm(c_i t^2 + d_i t + e_i)) \qquad (i = 1, \ldots, 28).$$

Note that, by [14, Proposition 4], 28 lines $L_i : x_i = a_i t + b_i$ are the 28 bitangents to $\mathscr{Q}$. As in §1, we denote the sections corresponding to P by s_P. Let $q_{z_o} \circ f_{\mathscr{Q},z_o} : S_{\mathscr{Q},z_o} \to \mathbb{P}^2$ be the map introduced in the Introduction and let $(q_{z_o} \circ f_{\mathscr{Q},z_o})^*(L_i) = s_i^+ + s_i^-$ $(i = 1, \dots, 28)$. Here, $s_i^+ = s_{P_i}$ and $s_i^- = s_{-P_i}$. Since $\Theta_{\infty,1} \cdot s_{\pm P_i} = 1$ and $O \cdot s_{\pm P_i} = 0$ $(i = 1, \dots, 28)$, by the explicit formula for the height pairing, we have the following lemma:

Lemma 3.1. *For P_i, $P_j \in \{\pm P_1, \dots, \pm P_{28}\}$,*

(i) *if $i = j$, then $\langle P_i, P_j \rangle = \frac{3}{2}$, $s_{P_i} \cdot s_{P_j} = -1$, and $s_{P_i} \cdot s_{-P_j} = 2$,*

(ii) *if $i \neq j$, then*

(a) *$\langle P_i, P_j \rangle = -\frac{1}{2}$ if and only if $s_{P_i} \cdot s_{P_j} = 1$ and $s_{P_i} \cdot s_{-P_j} = 0$,*

(b) *$\langle P_i, P_j \rangle = \frac{1}{2}$ if and only if $s_{P_i} \cdot s_{P_j} = 0$ and $s_{P_i} \cdot s_{-P_j} = 1$.*

Choose three distinct bitangents L_i, L_j, and L_k to $\mathscr{Q}$. Put $\triangle_{ijk} := L_i + L_j + L_k$. Then, by §1, we have connected numbers $c_{f_{\mathscr{Q}}}(\triangle_{ijk}) = 1$ or 2. From Lemma 3.1, we classify splitting types of three bitangents via the intersection number of $s_{\pm P_i}$'s. Let the matrix $G(i,j,k)$ be the matrix defined to be two times the Gramm matrix defined by the height pairing of P_i, P_j, and P_k. The diagonal entries of $G(i,j,k)$ are equal to 3, and the off-diagonal entries take values ± 1. Since $G(i,j,k)$ is a symmetric matrix, there are 8 possible choices of $G(i,j,k)$. By the following lemma, the 8 matrices are classified into two classes depending on $c_{f_{\mathscr{Q}}}(\triangle_{ijk})$.

Lemma 3.2. (i) *$c_{f_{\mathscr{Q}}}(\triangle_{ijk}) = 1$ if and only if*

$$G(i,j,k) \in \left\{ \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}, \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \right\}.$$

(ii) *$c_{f_{\mathscr{Q}}}(\triangle_{ijk}) = 2$ if and only if*

$$G(i,j,k) \in \left\{ \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \right\}.$$

PROOF. We give a proof when

$$G(i,j,k) = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix},$$

since the proof for the other 6 matrices can be done in the same manner.

(i) If

$$G(i,j,k) = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix},$$

i.e., $2\langle P_i, P_j\rangle = 2\langle P_j, P_k\rangle = 2\langle P_k, P_i\rangle = 1$, we have $s_{P_i} \cdot s_{P_j} = s_{P_j} \cdot s_{P_k} = s_{P_k} \cdot s_{P_i} = 0$ and $s_{P_i} \cdot s_{-P_j} = s_{P_j} \cdot s_{-P_k} = s_{P_k} \cdot s_{-P_i} = 1$ from Lemma 3.1. Since $\langle\, ,\rangle$ is symmetric, we obtain $s_{P_i} \cdot s_{-P_j} = s_{-P_j} \cdot s_{P_k} = s_{P_k} \cdot s_{-P_i} = s_{-P_i} \cdot s_{P_j} = s_{P_j} \cdot s_{-P_k} = s_{-P_k} \cdot s_{P_i} = 1$. This means that $c_{f_{\mathcal{Q}}}(\triangle_{ijk}) = 1$.

(ii) If

$$G(i,j,k) = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix},$$

i.e., $2\langle P_i, P_j\rangle = 2\langle P_j, P_k\rangle = 2\langle P_k, P_i\rangle = -1$, we have $s_{P_i} \cdot s_{P_j} = s_{P_j} \cdot s_{P_k} = s_{P_k} \cdot s_{P_i} = 1$ and $s_{P_i} \cdot s_{-P_j} = s_{P_j} \cdot s_{-P_k} = s_{P_k} \cdot s_{-P_i} = 0$ from Lemma 3.1. Hence we obtain $s_{P_i} \cdot s_{P_j} = s_{P_j} \cdot s_{P_k} = s_{P_k} \cdot s_{P_i} = 1$ and $s_{-P_i} \cdot s_{-P_j} = s_{-P_j} \cdot s_{-P_k} = s_{-P_k} \cdot s_{-P_i} = 1$, i.e., $c_{f_{\mathcal{Q}}}(\triangle_{ijk}) = 2$. □

The figures below explain configurations of $(q_{z_o} \circ f_{\mathcal{Q},z_o})^{-1}(\triangle_{ijk} \setminus \mathcal{Q})$ in case (i), (ii) from the proof of Lemma 3.2. Note that the preimages of points on $\mathcal{Q}$ are ignored. Also, as $z_o \notin \triangle_{ijk} \setminus \mathcal{Q}$, we infer that $c_{f_{\mathcal{Q}}}(\triangle_{ijk})$ is equal to the number of connected components of $(q_{z_o} \circ f_{\mathcal{Q},z_o})^{-1}(\triangle_{ijk} \setminus \mathcal{Q})$. Hence, by observing the matrices in the two classes above, we have the following lemmas:

LEMMA 3.3. *Let m_{ijk} be the number of upper-half entries of $G(i,j,k)$ taking values equal to -1. Under the above setting,*

(i) *$c_{f_{\mathcal{Q}}}(\triangle_{ijk}) = 1$ if and only if m_{ijk} is even,*

(ii) *$c_{f_{\mathcal{Q}}}(\triangle_{ijk}) = 2$ if and only if m_{ijk} is odd.*

We can restate the above Lemma in terms of determinants. Let I_3 be the identity matrix of size 3×3.

LEMMA 3.4. (i) *$c_{f_{\mathcal{Q}}}(\triangle_{ijk}) = 1$ if and only if $\det(G(i,j,k) - 3I_3) = 2$,*

(ii) *$c_{f_{\mathcal{Q}}}(\triangle_{ijk}) = 2$ if and only if $\det(G(i,j,k) - 3I_3) = -2$.*

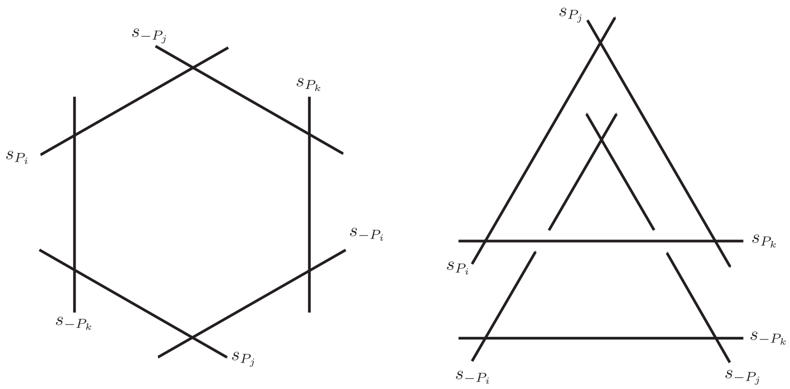

Fig. 1. Case (i), where $c_{f_{\mathcal{Q}}}(\triangle_{ijk}) = 1$, and case (ii), where $c_{f_{\mathcal{Q}}}(\triangle_{ijk}) = 2$.

3.2. The topology of plane quartic and its four bitangents via sub-arrangement. Let $\mathcal{Q}$ be a smooth plane quartic as in §3.1 and $L_1, \dots, L_{28}$ be 28 bitangents to $\mathcal{Q}$. Choose a subset $I \subset \{1, \dots, 28\}$ such that $\#I = 4$ and put $\mathcal{L}_I := \sum_{i \in I} L_i$. As in §3.1, we obtain the 4×4 matrix G_I which is defined as twice of the Gramm matrix defined by the height pairing of P_i's $(i \in I)$.

In order to consider the embedded topology of $\mathcal{Q} + \mathcal{L}_I$, we use the connected numbers of subarrangements $\underline{\mathrm{Sub}}_{\triangle}(\mathcal{Q}, \mathcal{L}_I)$. Let c_I be the map defined in §2.2 and put

$$m_I := \#\{\text{upper-half entries of } G_I \text{ equal to } -1\}$$

Since $\#\underline{\mathrm{Sub}}_{\triangle}(\mathcal{Q}, \mathcal{L}_I) = 4$, we have $\#c_I^{-1}(1) + \#c_I^{-1}(2) = 4$. Hence, there are 5 possible pairs

$$(\#c_I^{-1}(1), \#c_I^{-1}(2)) = (0,4), (1,3), (2,2), (3,1), (4,0).$$

By Proposition 2.2, it seems that a Zariski 5-ple may exist. However, the following Lemma shows that this is not true.

Lemma 3.5. *Under the above setting,* $(\#c_I^{-1}(1), \#c_I^{-1}(2)) = (0,4), (2,2), (4,0)$.

Proof. We claim that *there exists a non-negative integer M such that*

$$2m_I = 2M + \#c_I^{-1}(2).$$

In order to prove our claim, we consider the sum

$$\sum_{\{i_1, i_2, i_3\} \subset I} m_{i_1 i_2 i_3},$$

where $m_{i_1 i_2 i_3}$ is defined as in Lemma 3.3. Let us discribe this sum in two ways.

(I) Put $\triangle_{i_1 i_2 i_3} := L_{i_1} + L_{i_2} + L_{i_3}$. By Lemma 3.3, we have

$$c_{f_{\mathscr{Q}}}(\triangle_{i_1 i_2 i_3}) = 1 \qquad \text{if and only if } m_{i_1 i_2 i_3} = 0, 2,$$

$$c_{f_{\mathscr{Q}}}(\triangle_{i_1 i_2 i_3}) = 2 \qquad \text{if and only if } m_{i_1 i_2 i_3} = 1, 3.$$

We set $M_N := \#\{\triangle_{i_1 i_2 i_3} \mid m_{i_1 i_2 i_3} = N\}$ $(N = 0, 1, 2, 3)$. Then, we have

$$\begin{aligned}
\sum_{\{i_1, i_2, i_3\} \subset I} m_{i_1 i_2 i_3} &= \sum_{c_{f_{\mathscr{Q}}}(\triangle_{i_1 i_2 i_3}) = 1} m_{i_1 i_2 i_3} + \sum_{c_{f_{\mathscr{Q}}}(\triangle_{i_1 i_2 i_3}) = 2} m_{i_1 i_2 i_3} \\
&= 0 \cdot M_0 + 2 \cdot M_2 + 1 \cdot M_1 + 3 \cdot M_3 \\
&= 2(M_2 + M_3) + M_1 + M_3 \\
&= 2(M_2 + M_3) + \# c_I^{-1}(2).
\end{aligned}$$

Define M to be $M_2 + M_3$.

(II) When $m_I > 0$, fix an upper-half entry $g_{kl}(\{k, l\} \subset I)$ with value -1. Then, by the definitions of the matrices G_I and $G(i_1, i_2, i_3)$, we have

$$G(i_1, i_2, i_3) \text{ contains } g_{kl} \text{ as its entry if and only if } k, l \in \{i_1, i_2, i_3\}.$$

We may put $k = i_1$, $l = i_2$ without loss of generality. Then we have

$$\begin{aligned}
&\#\{G(i_1, i_2, i_3) \mid G(i_1, i_2, i_3) \text{ contains } g_{kl} \text{ as its entry}\} \\
&\quad = \#\{\{i_1, i_2, i_3\} \subset I \mid k = i_1,\ l = i_2\} \\
&\quad = 2.
\end{aligned}$$

Hence, by the definitions of m_I and $m_{i_1 i_2 i_3}$, we obtain

$$\sum_{\{i_1, i_2, i_3\} \subset I} m_{i_1 i_2 i_3} = 2 m_I.$$

If $m_I = 0$, $\sum_{\{i_1, i_2, i_3\} \subset I} m_{i_1 i_2 i_3}$ is also equal to zero. Thus, the above equation holds when $m_I = 0$.

Hence, $\# c_I^{-1}(2)$ must be an even number, which shows our statement. □

4. Examples

Let $\mathscr{Q}$ be the Klein quartic given by the affine equation:

$$F(t, x) := x^3 + t^3 x + t.$$

Then the generic fiber $E_{\mathscr{Q},z_o}$ of $\varphi_{\mathscr{Q},z_o}$ is $y^2 = F(t,x)$. From [14, Section 4], the 28 bitangents of $\mathscr{Q}$ are given by the following equations:

$$L_{0,j} : x_{0,j}(t) = -\zeta^j t - \zeta^{3j}, \qquad L_{1,j} : x_{1,j}(t) = -\zeta^j \varepsilon_1^2 t - \zeta^{3j}\varepsilon_3^{-2},$$

$$L_{2,j} : x_{2,j}(t) = -\zeta^j \varepsilon_2^2 t - \zeta^{3j}\varepsilon_1^{-2}, \qquad L_{3,j} : x_{3,j}(t) = -\zeta^j \varepsilon_3^2 t - \zeta^{3j}\varepsilon_2^{-2},$$

where $j = 0, \dots, 6$, $\zeta = e^{(2\pi i)/7}$, $\varepsilon_1 = \zeta + \zeta^{-1}$, $\varepsilon_2 = \zeta^2 + \zeta^{-2}$, $\varepsilon_3 = \zeta^4 + \zeta^{-4}$. Put

$$L_1 := L_{0,0}, \qquad L_2 := L_{1,0}, \qquad L_3 := L_{1,1}, \qquad L_4 := L_{3,3},$$

$$L_5 := L_{1,6}, \qquad L_6 := L_{3,4}, \qquad L_7 := L_{2,5}.$$

and rational points of $E_{\mathscr{Q},z_o}$ defined by $L_1, \dots, L_7$;

$$P_1 := (x_{0,0}(t), y_1(t)), \qquad P_2 := (x_{1,0}(t), y_2(t)), \qquad P_3 := (x_{1,1}(t), y_3(t)),$$

$$P_4 := (x_{3,3}(t), y_4(t)), \qquad P_5 := (x_{1,6}(t), y_5(t)), \qquad P_6 := (x_{3,4}(t), y_6(t)),$$

$$P_7 := (x_{2,5}(t), y_7(t)).$$

Here,

$$y_1(t) = \sqrt{-1}(t^2 + t + 1), \qquad y_2(t) = \sqrt{-1}\varepsilon_1(t^2 + a_1(\zeta)t + b_1(\zeta)),$$

$$y_3(t) = \sqrt{-1}\zeta^4\varepsilon_1(t^2 + \zeta^2 a_1(\zeta)t + \zeta^4 b_1(\zeta)),$$

$$y_4(t) = \sqrt{-1}\zeta^5\varepsilon_3(t^2 + a_3(\zeta)t + b_3(\zeta)),$$

$$y_5(t) = \sqrt{-1}\zeta^3\varepsilon_1(t^2 + \zeta^5 a_1(\zeta)t + \zeta^3 b_1(\zeta)),$$

$$y_6(t) = \sqrt{-1}\zeta^2\varepsilon_3(t^2 + \zeta^2 a_3(\zeta)t + \zeta^4 b_3(\zeta)),$$

$$y_7(t) = \sqrt{-1}\zeta^6\varepsilon_2(t^2 + (\zeta^2 + 2 + 2\zeta^6 + \zeta^4 + 4\zeta^3)t + \zeta^5 + 3\zeta^3 + 3\zeta^2 + 1 + 3\zeta^6),$$

where

$$a_1(\zeta) = 2\zeta^5 + \zeta^4 + \zeta^3 + 2\zeta^2 + 4, \qquad b_1(\zeta) = 3\zeta^5 + \zeta^4 + \zeta^3 + 3\zeta^2 + 3,$$

$$a_3(\zeta) = 2\zeta^5 + \zeta^4 + \zeta + 2 + 4\zeta^6, \qquad b_3(\zeta) = 3\zeta^4 + \zeta^3 + 1 + 3\zeta^6 + 3\zeta^5.$$

Note that for $L_1, \dots, L_7$, no three lines are concurrent and $\mathscr{Q} + \mathscr{L}_I$ $(I \subset \{1, \dots, 7\})$ all have the same combinatorics for fixed $\#I$.

- A Zariski pair for $\mathscr{Q}$ and its three bitangents

  We put

$$\mathscr{B}^1 := \mathscr{Q} + L_1 + L_2 + L_3, \qquad \mathscr{B}^2 := \mathscr{Q} + L_1 + L_2 + L_4.$$

For P_1 and P_2, consider sections s_{P_1} and s_{P_2} as curves in the affine part of $S_{\mathscr{Q},z_o}$ given by $s_{P_1} := (x_{0,0}(t), y_1(t))$ and $s_{P_2} := (x_{1,0}(t), y_2(t))$ with parameter t. Then $x_{0,0}(t) = x_{1,0}(t)$ and $y_1(t) = y_2(t)$ has a unique solution $t = \zeta + \zeta^{-1}$, which implies $s_{P_1} \cdot s_{P_2} = 1$. In the same way, we obtain $s_{P_2} \cdot s_{P_3} = s_{P_3} \cdot s_{P_1} = s_{P_1} \cdot s_{P_4} = 1$ and $s_{P_2} \cdot s_{P_4} = 0$. Hence, we have

$$G(1,2,3) = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \qquad G(1,2,4) = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}.$$

By Lemma 3.3, we have $c_{f_{\mathscr{Q}}}(\triangle_{123}) = 2$ and $c_{f_{\mathscr{Q}}}(\triangle_{124}) = 1$, then $(\mathscr{B}^1, \mathscr{B}^2)$ is a Zariski pair.

- <u>A Zariski triple for $\mathscr{Q}$ and its four bitangents</u>

We set $I_1 := \{1,2,3,5\}$, $I_2 := \{1,2,3,6\}$, $I_3 := \{1,2,4,7\}$ and put

$$\mathscr{B}^k := \mathscr{Q} + \mathscr{L}_{I_k} \qquad (k = 1,2,3).$$

As above, we have $G(1,2,3) = G(1,2,5) = G(1,3,5) = G(2,3,5)$, $G(1,2,4) = G(1,2,7) = G(1,4,7) = G(1,2,6) = G(1,3,6)$ and

$$G(2,3,6) = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \qquad G(2,4,7) = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

Hence, we obtain

$$(\#c_{I_1}^{-1}(1), \#c_{I_1}^{-1}(2)) = (4,0), \qquad (\#c_{I_2}^{-1}(1), \#c_{I_2}^{-1}(2)) = (2,2),$$
$$(\#c_{I_3}^{-1}(1), \#c_{I_3}^{-1}(2)) = (0,4).$$

By Proposition 2.2, $(\mathscr{B}^1, \mathscr{B}^2, \mathscr{B}^3)$ is a Zariski triple.

The existence of the above examples gives a proof to Theorem 1.1.

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