On the Integral Representation of Unbounded Self-Adjoint Transformations.

By

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Recently F. Riesz and E. R. Lorch[1] have given two proofs of the well-known formula of the integral representation of unbounded self-adjoint transformations in Hilbert space based on the theory of bounded transformations. I shall give here another proof of the formula according to their method.

1.

Let A be an unbounded self-adjoint transformation in Hilbert space $\mathfrak{H}$, and $\mathfrak{D}$ be its domain. F. Riesz and E. R. Lorch[1] have deduced the following results according to the J. v. Neumann's ingenious device: For any element $\mathfrak{h}$ of $\mathfrak{H}$ there determine uniquely two elements $\mathfrak{f}, \mathfrak{g}$ of $\mathfrak{H}$ and hold the relations

$$A\mathfrak{f}+\mathfrak{g} = \mathfrak{h},$$

and

$$\mathfrak{f}-A\mathfrak{g} = 0.$$

Put $\mathfrak{g} = B\mathfrak{h}$ and $\mathfrak{f} = C\mathfrak{h}$, then

(1) B and C are bounded self-adjoint transformations,

(2) $B\mathfrak{h} = 0$, when and only when $\mathfrak{h} = 0$,

(3) $BA = C$,

(4) $BC = CB$,

and

(5) $B^2+C^2 = B$.

2.

From sec. 1 (1) and (4) B and C are bounded self-adjoint and commutable, hence we can construct the complex resolution of the

(1) F. Riesz and E. R. Lorch, Trans. Amer. Math. Soc., **39** (1936), 331–340.

identity from those corresponding to B and C.[1] Denoting such resolution of the identity by $E(U)$,[2] we can write B and C in the forms[3]

$$B\mathfrak{h} = \int_R x dE(U)\mathfrak{h}$$

$$C\mathfrak{h} = \int_R y dE(U)\mathfrak{h}$$

where R is the xy-plane, and $\mathfrak{h}$ is any element of $\mathfrak{H}$.

From sec. 1 (5) we have

$$\int_R (x^2+y^2-x)dE(U)\mathfrak{h} = 0$$

therefore $E(U) \equiv 0$ except on the circle K of radius $\frac{1}{2}$ with center $\left(\frac{1}{2}, 0\right)$ on xy-plane. And from sec. 1 (2) $E(U) \equiv 0$ on the y-axis. Hence we have

$$E(K_0) = 1 \tag{1}$$

where K_0 is the set of points on K except the origin.

From sec. 1 (3), for any element $\mathfrak{f}$ of $\mathfrak{D}$ we have

$$\int_{K_0} x dE(U)A\mathfrak{f} = \int_{K_0} y dE(U)\mathfrak{f}. \tag{2}$$

Let K_ε be the set of points on K_0 except on the ε neighbourhood of the origin, then since $\frac{1}{x}$ is bounded on K_ε, $\int_{K_\varepsilon} \frac{1}{x} dE(U)$ has meaning for any element of $\mathfrak{H}$. Therefore we have from (2)

$$\int_{K_\varepsilon} \frac{1}{x} dE(U) \int_{K_0} x dE(U)A\mathfrak{f} = \int_{K_\varepsilon} \frac{1}{x} dE(U) \int_{K_0} y dE(U)\mathfrak{f}$$

(1) J. v. Neumann, Math. Ann., **102** (1929), 411. T. Ogasawara, this journal, **5** (1935), 125–126.

(2) $E(U)$ is the resolution of the identity depending on the Borel sets U of xy-plane, that is $E(R)=1$, R being xy-plane, $E(U)E(U')=E(UU')$, and $E(U)=E(U_1)+E(U_2)+\ldots$, when $U=U_1+U_2+\ldots$. Cf. F. Maeda, this journal, **4** (1934), 73. And for such construction see T. Ogasawara, *loc. cit.* 125–126.

(3) T. Ogasawara, *loc. cit.* 128. Here the integration means that with respect to the vector valued set function. Cf. F. Maeda, *loc. cit.* 60–69.

that is,
$$E(K_\varepsilon)A\mathfrak{f} = \int_{K_\varepsilon} \frac{y}{x} dE(U)\mathfrak{f}.$$

Tending ε to 0 we have

$$E(K_0)A\mathfrak{f} = \int_{K_0} \frac{y}{x} dE(U)\mathfrak{f},$$

since from (1) $E(K_0) = 1$, therefore

$$A\mathfrak{f} = \int_{K_0} \frac{y}{x} dE(U)\mathfrak{f} \tag{3}$$

Let $\lambda = \frac{y}{x}$, then this expresses the transformation between K_0 and λ-axis. Let U' be any Borel set on the λ-axis, and U be its transform on K_0. And put

$$E'(U') = E(U)$$

then it will easily be seen that $E'(U')$ is the resolution of the identity on the λ-axis, and (3) will become

$$A\mathfrak{f} = \int_{R_1} \lambda dE'(U')\mathfrak{f}$$

where R_1 is the real λ-axis.

As is well known, $\int_{R_1} \lambda dE'(U')\mathfrak{f}$ is symmetric, and since it is an extension of A, therefore we obtain

$$A = \int_{R_1} \lambda dE'(U')$$