On Some Solutions of $\frac{\sqrt{\Delta}}{2}\epsilon_{stpq}K_{lm}^{\cdot\cdot pq} = K_{lmst}$.

By

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The object of this paper is to find out from the physical standpoint some useful solutions of the fundamental equation for g_{ij}:

$$\frac{\sqrt{\Delta}}{2}\epsilon_{stpq}K_{lm}^{\cdot\cdot pq} = K_{lmst}. \tag{0.1}$$

The general solutions of this equation have been obtained by T. Sibata and one of the present authors,[1] but these are not yet directly applicable to the physical problem, so we shall investigate this problem from another point of view. First, we shall find out some approximate solutions, and then proceed to the finite solutions.

§ 1. Approximate solutions.

If we exclude the euclidean terms, the most general approximate solution of the equation (0.1) is given by solving the following equation[2]:

$$\left.\begin{array}{l} g_{ij} = \delta_{ij} + h_{ij}, \qquad |h_{ij}|^2 \ll 1. \\ -\frac{\partial h_{1m}}{\partial x^2} + \frac{\partial h_{2m}}{\partial x^1} + \frac{\partial h_{3m}}{\partial x^4} - \frac{\partial h_{4m}}{\partial x^3} = 0 \\ \frac{\partial h_{1m}}{\partial x^3} + \frac{\partial h_{2m}}{\partial x^4} - \frac{\partial h_{3m}}{\partial x^1} - \frac{\partial h_{4m}}{\partial x^2} = 0 \\ \frac{\partial h_{1m}}{\partial x^4} - \frac{\partial h_{2m}}{\partial x^3} + \frac{\partial h_{3m}}{\partial x^2} - \frac{\partial h_{4m}}{\partial x^1} = 0 \end{array}\right\} (i, j, \ldots, l, m, \ldots = 1, 2, 3, 4) \tag{1.1}$$

(1) T. Sibata and K. Morinaga, This Journal, **6** (1935), 173.

(2) T. Sibata, This Journal, **5** (1935), 195. The corresponding equation for $g_{ij} = \bar{\delta}_{ij} + k_{ij}$ can be obtained by the transformation $x^i = \bar{x}^i$ $(i = 1, 2, 3)$, $x^4 = i\bar{x}^4$ where $\bar{\delta}_{ij} = 0$ for $i \neq j$ and $\bar{\delta}_{11} = \bar{\delta}_{22} = \bar{\delta}_{33} = -\bar{\delta}_{44} = 1$.

The solution to be considered first is the orthogonal one (i. e. $g_{ij}=0$ for $i \neq j$), but as is readily seen from (1.1), the solution of this type must have the forms:

$$g_{11}=f_1(x^1)\,,\qquad g_{22}=f_2(x^2)\,,\qquad g_{33}=f_3(x^3)\,,\qquad g_{44}=f_4(x^4)\,,$$

which supplies zero curvature tensors. Hence we know that a non-euclidean orthogonal solution can not exist.

Therefore, let us consider another simple and useful case.

It is always possible to choose the coordinate system so that one of the congruences of parametric curves may be orthogonal to the others. We will take such a congruence as time-coordinate $x^4\ (=it)$ as in the case of the ordinary relativity. Now let us assume that the gravitational phenomenon occurs with a special direction (for example, a gravitating particle is spinning in this direction). We take this direction as x^3-axis and assume that this direction is orthogonal to x^1- and x^2-axes.

From these assumption, we have

$$g_{14}=g_{24}=g_{34}=g_{13}=g_{23}=0\,, \tag{1.2}$$

and then from (1.1) we may take $g_{33}=1$, $g_{44}=1$. From (1.1) also we have

$$\frac{\partial h_{11}}{\partial x^2}=\frac{\partial h_{12}}{\partial x^1} \tag{1.3a}$$

$$\frac{\partial h_{11}}{\partial x^3}=-\frac{\partial h_{12}}{\partial x^4} \tag{1.3b}$$

$$\frac{\partial h_{11}}{\partial x^4}=\frac{\partial h_{12}}{\partial x^3} \tag{1.3c}$$

$$\frac{\partial h_{22}}{\partial x^1}=\frac{\partial h_{12}}{\partial x^2} \tag{1.4a}$$

$$\frac{\partial h_{22}}{\partial x^4}=-\frac{\partial h_{12}}{\partial x^3} \tag{1.4b}$$

$$\frac{\partial h_{22}}{\partial x^3}=\frac{\partial x_{12}}{\partial x^4} \tag{1.4c}$$

from which we have the general solution(1):

$$\left.\begin{aligned} h_{11} &= f_1(X, Y)+f_2(\overline{X}, \overline{Y})+\int\frac{\partial m(x^1, x^2)}{\partial x^1}dx^2+n_1(x^1) \\ h_{12} &= if_1(X, Y)-if_2(\overline{X}, \overline{Y})+m(x^1, x^2) \\ h_{22} &= -f_1(X, Y)-f_2(\overline{X}, \overline{Y})+\int\frac{\partial m(x^1, x^2)}{\partial x^2}dx^1+n_2(x^2) \end{aligned}\right\} \quad (1.8)$$

where f's, m, and n's are arbitrary functions and

$$X = x^3+ix^4, \quad \overline{X} = x^3-ix^4; \quad Y = x^1+ix^2, \quad \overline{Y} = x^1-ix^2.$$

But as is readily seen m-terms and n-terms give euclidean metric. Hence if we exclude euclidean terms, the most general approximate solutions of the equation (0.1), which satisfy (1.2) are given by

$$\left.\begin{aligned} g_{11} &= 1-i\varphi(X, Y)+i\chi(\overline{X}, \overline{Y}) \\ g_{12} &= \varphi(X, Y)+\chi(\overline{X}, \overline{Y}) \\ g_{22} &= 1+i\varphi(X, Y)-i\chi(\overline{X}, \overline{Y}) \\ g_{3m} &= \delta_{3m}, \quad g_{4m} = \delta_{4m} \end{aligned}\right\} \quad (A)$$

where φ and χ are arbitrary functions satisfying the relation

$$|\varphi|^2 \ll 1, \quad |\chi|^2 \ll 1, \quad |\varphi\chi| \ll 1. \quad (1.9)$$

(1) From (1.3b), (1.4c), (1.3c) and (1.4b) we have

$$h_{22} = a(x^1, x^2)-h_{11}, \quad (a\text{: arbitrary function}) \quad (1.5)$$

and

$$\left(\frac{\partial^2}{\partial x^3\partial x^3}+\frac{\partial^2}{\partial x^4\partial x^4}\right)h_{11} = 0, \quad \left(\frac{\partial^2}{\partial x^3\partial x^3}+\frac{\partial^2}{\partial x^4\partial x^4}\right)h_{12} = 0,$$

hence,

$$\left.\begin{aligned} h_{11} &= b_1(X, x^1, x^2)+b_2(\overline{X}, x^1, x^2) \\ h_{12} &= c_1(X, x^1, x^2)+c_2(\overline{X}, x^1, x^2) \end{aligned}\right\}. \quad (1.6)$$

where b's and c's are arbitrary functions and $X = x^3+ix^4$, $\overline{X} = x^3-ix^4$. By substituting (1.5) and (1.6) into (1.3b) and (1.3c) we have

$$c_1 = ib_1+d_1(x^1, x^2), \quad c_2 = -ib_2+d_2(x^1, x^2),$$

hence,

$$h_{12} = ib_1-ib_2+e(x^1, x^2). \quad (e\text{: arbitrary function}) \quad (1.7)$$

If we substitute (1.6) and (1.7) into (1.3a) and (1.4a) we have

$$b_1 = f_1(X, Y)+m_1(Y, \overline{Y}), \quad b_2 = f_2(\overline{X}, \overline{Y})+m_2(Y, \overline{Y}),$$

where $Y = x^1+ix^2$, $\overline{Y} = x^1-ix^2$, consequently we have

$$h_{22} = -f_1-f_2+m_3(Y, \overline{Y}).$$

Substituting these results into (1.3a) and (1.4a) again, we obtain (1.8).

§ 2. Finite solutions.

In this section we see when the approximate solutions above obtained may also be finite solutions. Removing the condition (1.9) and from (A) we have

$$\left.\begin{aligned} \varDelta &= |\, g_{ij} \,| = 1-4\varphi\chi \\ g^{11} &= \frac{1}{\varDelta}\{1+i(\varphi-\chi)\} \\ g^{12} &= -\frac{1}{\varDelta}(\varphi+\chi) \\ g^{22} &= \frac{1}{\varDelta}\{1-i(\varphi-\chi)\} \\ g^{3m} &= \delta^{3m}, \qquad g^{4m} = \delta^{4m} \end{aligned}\right\} \tag{2.1}$$

Calculating the values of curvature tensors K_{ijlm}, which for convenience' sake we express in the form:

$$K_{ijlm} = P_{ijlm} + \frac{1}{2} Q_{ijlm},$$

where $$P_{ijlm} = -\left\{ {h \atop li} \right\}[mj, h] + \left\{ {h \atop lj} \right\}[mi, h],$$

$$Q_{ijlm} = \frac{\partial^2 g_{im}}{\partial x^j \partial x^l} + \frac{\partial^2 g_{jl}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^m} - \frac{\partial^2 g_{jm}}{\partial x^i \partial x^l},$$

we have

Table of P_{ijlm}.

ij \ lm	12	13	14	23	24	34
12	$2\left(\frac{1}{\varDelta}\beta_1\beta_2+a_1a_2\right)$					
13	$\frac{i}{\varDelta}\left(a_2\beta_1-a_1\beta_2\right)$	$\frac{1}{\varDelta}\left\{a_1a_2-i(\varepsilon_1-\varepsilon_2)\right\}$				
14	$\frac{1}{\varDelta}\left(a_2\beta_1+a_1\beta_2\right)$	$-\frac{1}{\varDelta}(\varepsilon_1+\varepsilon_2)$	$\frac{1}{\varDelta}\left\{a_1a_2+i(\varepsilon_1-\varepsilon_2)\right\}$			
23	$-\frac{1}{\varDelta}\left(a_2\beta_1+a_1\beta_2\right)$	$-\frac{1}{\varDelta}(\varepsilon_1+\varepsilon_2)$	$\frac{1}{\varDelta}\left\{-a_1a_2+i(\varepsilon_1-\varepsilon_2)\right\}$	$\frac{1}{\varDelta}\left\{a_1a_2+i(\varepsilon_1-\varepsilon_2)\right\}$		
24	$\frac{i}{\varDelta}\left(a_2\beta_1-a_1\beta_2\right)$	$\frac{1}{\varDelta}\left\{a_1a_2+i(\varepsilon_1-\varepsilon 2)\right\}$	$\frac{1}{\varDelta}(\varepsilon_1+\varepsilon_2)$	$\frac{1}{\varDelta}(\varepsilon_1+\varepsilon_2)$	$\frac{1}{\varDelta}\left\{a_1a_2-i(\varepsilon_1-\varepsilon_2)\right\}$	
34	$\frac{2}{\varDelta}a_1a_2$	0	0	0	0	0

$(P_{ijlm} = P_{lmij} = -P_{ijml} = -P_{jilm})$

where

$$\alpha_1 = \frac{\partial \varphi}{\partial X}, \quad \alpha_2 = \frac{\partial \chi}{\partial \overline{X}}, \quad \beta_1 = \frac{\partial \varphi}{\partial Y}, \quad \beta_2 = \frac{\partial \chi}{\partial \overline{Y}}, \quad \varepsilon_1 = \varphi\left(\frac{\partial \chi}{\partial \overline{X}}\right)^2, \quad \varepsilon_2 = \chi\left(\frac{\partial \varphi}{\partial X}\right)^2,$$

Table of Q_{ijlm}.

ij \ lm	13	14	23	24
13	$i(\gamma_1-\gamma_2)$			
14	$-(\gamma_1+\gamma_2)$	$-i(\gamma_1-\gamma_2)$		
23	$-(\gamma_1+\gamma_2)$	$-i(\gamma_1-\gamma_2)$	$-i(\gamma_1-\gamma_2)$	
24	$-i(\gamma_1-\gamma_2)$	$(\gamma_1+\gamma_2)$	$(\gamma_1+\gamma_2)$	$i(\gamma_1-\gamma_2)$

$(Q_{ijlm} = Q_{lmij} = -Q_{ijml} = -Q_{jilm})$

$Q_{ij12} = Q_{ij34} = 0$

where $\qquad \gamma_1 = \dfrac{\partial^2 \varphi}{\partial X^2}, \qquad \gamma_2 = \dfrac{\partial^2 \chi}{\partial \overline{X}^2},$

Since in our case the equation (0.1) becomes as follows

$$\sqrt{\Delta}\, K_{st34} = K_{st12} \tag{2.2a}$$

$$(\sqrt{\Delta}\, K_{st24} + K_{st13}) + i(\varphi - \chi)K_{st13} - (\varphi + \chi)K_{st23} = 0 \tag{2.2b}$$

$$(-\sqrt{\Delta}\, K_{st23} + K_{st14}) + i(\varphi - \chi)K_{st14} - (\varphi + \chi)K_{st24} = 0, \tag{2.2c}$$

substituting the actual values of K_{ijlm} into (2.2a) we have

$$\alpha_1\alpha_2(\sqrt{\Delta} - \Delta) = \beta_1\beta_2, \quad \alpha_2\beta_1 - \alpha_1\beta_2 = 0, \quad \alpha_2\beta_1 + \alpha_1\beta_2 = 0 \quad \alpha_1\alpha_2 = 0,$$

$$\therefore \qquad \alpha_1\alpha_2 = \beta_1\beta_2 = \alpha_2\beta_1 = \alpha_1\beta_2 = 0.$$

Conversely, if these relations hold, (2.2a) is satisfied identically, so the following two cases are possible.

Case I. When $\alpha_2 = \beta_2 = 0$. Case II. When $\alpha_1 = \beta_1 = 0$.

In Case I we see that $\gamma_2 = \varepsilon_1 = 0$ and χ becomes a constant, and accordingly K_{ijlm} is given by

Table of K_{ijlm}.

ij \ lm	13	14	23	24
13	ik			
14	$-k$	$-ik$		
23	$-k$	$-ik$	$-ik$	
24	$-ik$	k	k	ik

$$K_{ij12} = K_{ij34} = 0$$

where $k = \frac{1}{2}\gamma_1 + \frac{1}{\varDelta}\varepsilon_2$

$$= \frac{1}{2}\frac{\partial^2\varphi}{\partial X^2} + \frac{1}{\varDelta}\chi\left(\frac{\partial\varphi}{\partial X}\right)^2 .$$

Substituting these values into (2.2b) and (2.2c), we have only one equation[1]:

$$1 - \sqrt{\varDelta} = 2i\chi . \tag{2.3}$$

Hence from (2.1), (2.3) and taking account of $\chi = \text{constant}$, we have

$$4\varphi\chi = \text{constant}.$$

If $\chi \neq 0$, it must be that $\varphi = \text{constant}$ and curvature tensor vanishes, therefore we must have

$$\chi = 0 .$$

Conversely if $\chi = 0$, (2.3) is satisfied, consequently (A) is a finite solution of the equation (0.1).

In Case II, by the same way as the above we can get

$$\varphi = 0 .$$

So we have the two types of finite solutions:

$$(F_1):\ \chi = 0 , \qquad (F_2):\ \varphi = 0 .$$

Hence we have the result: *When the space differs infinitesimally from euclidean under the assumption (1.2), the solutions of the equation (0.1) are given by (A) i.e.* $(F_1)+(F_2)$, *but in the finite case the solution breaks up into either* (F_1) *or* (F_2).

In the coordinate system $x = x^1$, $y = x^2$, $z = x^3$, $t = -ix^4$, (F_1) and (F_2) are written as follows:

(1) We assume that $k \neq 0$, otherwise the space becomes euclidean.

$$(F_1):\begin{cases} g_{11} = 1-i\varphi(z-t, x+iy) \\ g_{12} = \varphi(z-t, x+iy) \\ g_{22} = 1+i\varphi(z-t, x+iy) \\ g_{3m} = \delta_{3m}, \quad g_{4m} = -\delta_{4m} \end{cases} \qquad (F_2):\begin{cases} g_{11} = 1+i\chi(z+t, x-iy) \\ g_{12} = \chi(z+t, x-iy) \\ g_{22} = 1-i\chi(z+t, x-iy) \\ g_{3m} = \delta_{3m}, \quad g_{4m} = -\delta_{4m} \end{cases}$$

from which we know that (F_1) or (F_2) has wave properties which propagates in + direction or − direction of z-axis respectively. The physical interpretations of this result will be found in Mimura and Iwatsuki's paper.(1)

§ 3. Some properties of the solutions (F_1) and (F_2).

Concerning the finite solutions (F_1) and (F_2), we can prove the following theorem.

Both (F_1) and (F_2) can not be transformed into the orthogonal form:

$$'g_{ij} = 0 \qquad (i \neq j) \tag{3.1}$$

by any real coordinate transformation of x, y, z, t.

Proof. We use x^1, x^2, x^3, x^4 for the coordinates x, y, z, t. A real transformation is given by

$$\left.\begin{aligned} x^i &= f^i('x^1, 'x^2, 'x^3, 'x^4), \\ Q^i_j &= \frac{\partial x^i}{\partial' x^j}, \qquad |Q^i_j| \neq 0. \end{aligned}\right\} \tag{3.2}$$

We will prove the theorem in the case of the solution (F_1). By the transformation (3.2), g_{ij} is transformed by the relation

$$\begin{aligned} 'g_{ij} &= Q^l_i Q^m_j g_{lm} \\ &= \bar{\delta}_{lm} Q^l_i Q^m_j - i\varphi\{(Q^1_i Q^1_j - Q^2_i Q^2_j) + i(Q^1_i Q^2_j + Q^2_i Q^1_j)\}, \end{aligned} \tag{3.3}$$

where $\bar{\delta}_{lm} = 0$ for $l \neq m$ and $\bar{\delta}_{11} = \bar{\delta}_{22} = \bar{\delta}_{33} = -\bar{\delta}_{44} = 1$. If we put

$$\varphi(X, Y) = a+ib,$$

where a and b are real functions, we have $a^2+b^2 \neq 0$.

(1) Y. Mimura and T. Iwatsuki, This Journal, **6** (1935), 203.

From (3.1) we have

$$\bar{\delta}_{lm}Q_i^lQ_j^m+b(Q_i^1Q_j^1-Q_i^2Q_j^2)+a(Q_i^1Q_j^2+Q_i^2Q_j^1)=0, \quad (i\neq j) \quad (3.4a)$$

$$a(Q_i^1Q_j^1-Q_i^2Q_j^2)-b(Q_i^1Q_j^2+Q_i^2Q_j^1)=0. \quad (i\neq j) \quad (3.4b)$$

(3.4b) can be rewritten in the form:

$$Q_i^1(aQ_j^1-bQ_j^2)-Q_i^2(bQ_j^1+aQ_j^2)=0. \quad (i\neq j) \quad (3.5)$$

The cases are separated in the following five.

Case I, when $aQ_i^1-bQ_i^2\neq 0$. From (3.5) we easily see that $|Q_j^i|=0$, which contradicts the assumption (3.2).

Case II, when only one of $aQ_i^1-bQ_i^2$ vanishes, say,

$$aQ_1^1-bQ_1^2=0, \qquad aQ_i^1-bQ_i^2\neq 0. \qquad (i=2,3,4) \quad (3.6)$$

When $bQ_1^1+aQ_1^2\neq 0$, from (3.5) we have

$$Q_a^1=Q_a^2=0, \qquad (a=2,3,4)$$

from which we get $|Q_j^i|=0$.

When $bQ_1^1+aQ_1^2=0$, from (3.5) and (3.6) we have

$$Q_1^1=Q_1^2=0,$$

$$\begin{vmatrix} Q_3^1 & Q_3^2 \\ Q_4^1 & Q_4^2 \end{vmatrix} = \begin{vmatrix} Q_2^1 & Q_2^2 \\ Q_4^1 & Q_4^2 \end{vmatrix} = \begin{vmatrix} Q_2^1 & Q_2^2 \\ Q_3^1 & Q_3^2 \end{vmatrix} = 0,$$

so we get $|Q_j^i|=0$.

Case III, when only two of $aQ_i^1-bQ_i^2$ vanish, say,

$$aQ_i^1-bQ_i^2=0, \qquad (i=1,2) \text{ and } \neq 0, \ (i=3,4)$$

in the same way as the Case II, we have

$$Q_1^1=Q_1^2=Q_2^1=Q_2^2=0,$$

hence from the condition $|Q_j^i|\neq 0$, it must be that

$$\begin{vmatrix} Q_1^3 & Q_2^3 \\ Q_1^4 & Q_2^4 \end{vmatrix} \neq 0. \qquad (3.7)$$

On the other hand from (3.4a) we have

$$\begin{cases} Q_1^3Q_3^3 - Q_1^4Q_3^4 = 0\,, \\ Q_2^3Q_3^3 - Q_2^4Q_3^4 = 0\,, \end{cases} \qquad \begin{cases} Q_1^3Q_4^3 - Q_1^4Q_4^4 = 0\,, \\ Q_2^3Q_4^3 - Q_2^4Q_4^4 = 0\,, \end{cases}$$

therefore using (3.7),

$$Q_3^3 = Q_3^4 = 0\,, \qquad Q_4^3 = Q_4^4 = 0\,.$$

So we have $x^3 = x^3('x^1, 'x^2)$, $x^4 = x^4('x^1, 'x^2)$, and therefore from (3.3) it must be that $'g_{11} = 'g_{11}('x^1, 'x^2)$, $'g_{22} = 'g_{22}('x^1, 'x^2)$, consequently using the lemma which will be stated at the end of this section, we can deduce that $K_{ijlm} = 0$.

Case IV, when three of $aQ_i^1 - bQ_i^2$ vanish, say,

$$aQ_i^1 - bQ_i^2 = 0\,, \qquad (i = 1, 2, 3) \text{ and } \neq 0, \ (i = 4).$$

After a little calculation, we see that this case can be reduced to the type of Case III.

Case V, when $aQ_i^1 - bQ_i^2 = 0$, $(i = 1, 2, 3, 4)$. In this case it is easily proved that $|Q_j^i| = 0$.

Putting together all the results above obtained we see that the theorem is true.

From this theorem we can also deduce that: *By any real coordinate transformation both* (F_1) *and* (F_2) *can not be transformed into such form that three elements of all the matrices* $\begin{pmatrix} g_{1m} \\ g_{2m} \\ g_{3m} \\ g_{4m} \end{pmatrix}$ $(m = 1, 2, 3, 4)$ *vanish.*

Proof. For such g's we can find in each case a simple real transformation by which the matrix (g_{ij}) is transformed into the orthogonal form.(1) Therefore by the theorem above the proposition is true.

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(1) For example, $\begin{pmatrix} 0 & \times & 0 & 0 \\ \times & 0 & 0 & 0 \\ 0 & 0 & 0 & \times \\ 0 & 0 & \times & 0 \end{pmatrix}$ is transformed into the orthogonal form by the transformation

$$x^1 = 'x^1 + 'x^2\,, \quad x^2 = 'x^1 - 'x^2\,, \quad x^3 = 'x^3 + 'x^4\,, \quad x^4 = 'x^3 - 'x^4\,,$$

where $(\times)$ denotes non-zero element.

Next we will prove the following lemma used in the proof of the first theorem of this section, i. e.

If g_{ij} *of the form*

$$g_{ij}=0 \quad (i \neq j)\,, \qquad g_{11}=g_{11}(x^1, x^2)\,, \qquad g_{22}=g_{22}(x^1, x^2)\,,$$

satisfies the equation (*0.1*), K_{ijlm} *must vanish.*

Proof. Putting

$$g_{ii}=H_i^2\,, \qquad (i: \text{not summed}) \tag{3.8}$$

the equation (0.1) is written down in the form

$$\frac{K_{st12}}{H_1H_2}=\epsilon_1\frac{K_{st34}}{H_3H_4}\,, \quad \frac{K_{st13}}{H_1H_3}=\epsilon_2\frac{K_{st24}}{H_2H_4}\,, \quad \frac{K_{st14}}{H_1H_4}=\epsilon_3\frac{K_{st23}}{H_2H_3}\,, \tag{3.9}$$

where ϵ's are constants whose values are $+1$ or -1. But from (3.8) we have

$$K_{hijk}=0\,, \qquad (h, i, j, k \neq) \tag{3.10a}$$

$$K_{hiik}=H_i\left(\frac{\partial^2 H_i}{\partial x^h \partial x^k}-\frac{\partial H_i}{\partial x^h}\frac{\partial \log H_h}{\partial x^k}-\frac{\partial H_i}{\partial x^k}\frac{\partial \log H_k}{\partial x^h}\right), \quad (h, i, k \neq) \tag{3.10b}$$

where i is not summed. Hence from (3.9) and (3.10a)

$$\frac{K_{hiih}}{H_iH_h}=\epsilon\frac{K_{hijk}}{H_jH_k}=0\,. \quad (h, i, j, k \neq\,;\ h, i: \text{not summed}) \tag{3.11}$$

And from (3.8) and (3.10b),

$$K_{2113}=K_{2114}=K_{3114}=0\,.$$

Hence from the last equations and (3.9), we have

$$\left.\begin{aligned} K_{1213}=K_{1224}=K_{3413}=K_{3424}=0 \\ K_{1214}=K_{1223}=K_{3414}=K_{3423}=0 \\ K_{1314}=K_{1323}=K_{2414}=K_{2423}=0 \end{aligned}\right\} \tag{3.12}$$

So that from (3.10a), (3.11) and (3.12), we have $K_{ijlm}=0$. So we have proved the lemma.

From this lemma we can easily see that the equation (0.1) has no non-flat solution of Schwarzschild's type.

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