Complete and Simpler Treatment of Wave Geometry.

By

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§ 1. Introduction.

In wave geometry[1] we have defined the expression for the metric in general microscopic space by

$$ds\psi = dx^i\gamma_i\psi$$

where γ's are $4-4$ matrices satisfying the equation

$$\gamma_{(i}\gamma_{j)} = g_{ij}I$$

and ψ is a $1-4$ matrix given as a solution of the "unknown Dirac equation." And we have investigated the transformations and parallel displacement which make $ds\psi = 0$ invariant. In this paper, from another point of view we shall consider the parallel displacements and the differential equations for ψ obtained by this parallelism.

§ 2. Vectors which satisfy the relation $\rho^i\gamma_i\psi = 0$.

First, we will show that there exists a vector ρ^i satisfying the relation:

$$\rho^i\gamma_i\psi = 0. \tag{2.1}$$

Since γ_i are expressed as

$$\gamma_i = Uh^j_i\overset{\circ}{\gamma}_jU^{-1} \tag{2.2}$$

where h^i_j are defined by

$$g_{ij} = \sum_{l=1}^{4} h^l_i h^l_j,$$

(1) K. Morinaga, Wave Geometry, This Journal, **5** (1935), 151.

if we put $$\rho^i h_i^j = p^j \tag{2.3}$$

and $$U^{-1}\psi = \overline{\psi},$$

(2.1) is rewritten as $$Up^j \overset{\circ}{\gamma}_j \overline{\psi} = 0$$

or $$p^j \overset{\circ}{\gamma}_j \overline{\psi} = 0 \tag{2.4}$$

Rewriting (2.4) in actual form, we have

$$\left.\begin{aligned}
&p^1\overline{\psi}_3+p^2(-i\overline{\psi}_4)+p^3\overline{\psi}_4+p^4(-i\overline{\psi}_3) = 0\\
&p^1\overline{\psi}_4+p^2(-i\overline{\psi}_3)+p^3(-\overline{\psi}_3)+p^4(i\overline{\psi}_4) = 0\\
&p^1\overline{\psi}_1+p^2(i\overline{\psi}_2)+p^3(-\overline{\psi}_2)+p^4(i\overline{\psi}_1) = 0\\
&p^1\overline{\psi}_2+p^2(i\overline{\psi}_1)+p^3(\overline{\psi}_1)+p^4(-i\overline{\psi}_2) = 0
\end{aligned}\right\} \tag{2.5}$$

It is easily seen that the rank of the matrix formed by the coefficients of $p^1, \ldots, p^4$ in (2.5):

$$\begin{vmatrix}
\overline{\psi}_3 & -i\overline{\psi}_4 & \overline{\psi}_4 & -i\overline{\psi}_3\\
\overline{\psi}_4 & -i\overline{\psi}_3 & -\overline{\psi}_3 & i\overline{\psi}_4\\
\overline{\psi}_1 & i\overline{\psi}_2 & -\overline{\psi}_2 & i\overline{\psi}_1\\
\overline{\psi}_2 & i\overline{\psi}_1 & \overline{\psi}_1 & -i\overline{\psi}_2
\end{vmatrix} \tag{2.6}$$

is 3 at most. Therefore there exists a set $p^1, \ldots, p^4$ at least which satisfies (2.5), namely ρ^i satisfying the relation (2.1).

When the rank of matrix (2.6) is just *3*, there exists a vector ρ^i which satisfies (2.1). When the rank is *2*, there exist two independent vectors $\rho_a^i (a = 1, 2)$ which satisfy (2.1). And from the form of (2.6) we see that the case when the rank of (2.6) is *1* does not occur. So we shall investigate the problem only in the two cases when the rank of (2.6) is *3* or *2*.

For the sake of convenience and future consideration, we rewrite (2.5) in the following form:

$$\left.\begin{aligned}
&(p^1-ip^4)\psi_3+(p^3-ip^2)\psi_4 = 0\\
&-(p^3+ip^2)\psi_3+(p^1+ip^4)\psi_4 = 0\\
&(p^1+ip^4)\psi_1-(p^3-ip^2)\psi_2 = 0\\
&(p^3+ip^2)\psi_1+(p^1-ip^4)\psi_2 = 0
\end{aligned}\right\} \tag{2.7}$$

(I) When the rank of the matrix (2.6) is *3*. From (2.7), we know that in order that there may exist nonvanishing ψ, the determinant formed by the coefficients of ψ's must vanish, i. e.

$$\sum_{i=1}^{4} p^i p^i = 0 \tag{2.8}$$

or, using (2.3),

$$g_{ij}\rho^i\rho^j = 0 . \tag{2.9}$$

(II) When the rank of (2.6) is *2*. There exist two independent solutions, say $p_a^1, \ldots, p_a^4$ $(a = 1, 2)$, which satisfy (2.5). Substituting these p_a^i and eliminating $\psi_1, \ldots, \psi_4$, we have

$$\left.\begin{aligned}
&\sum_{i=1}^{4} p_1^i p_1^i = 0 \qquad \sum_{i=1}^{4} p_2^i p_2^i = 0 \\
&\sum_{i=1}^{4} p_1^i p_2^i + 2i\{p_1^{[2}p_2^{3]} + p_1^{[1}p_2^{4]}\} = 0 \\
&\sum_{i=1}^{4} p_1^i p_2^i - 2i\{p_1^{[2}p_2^{3]} + p_1^{[1}p_2^{4]}\} = 0 \\
&\{p_1^{[3}p_2^{1]} - p_1^{[4}p_2^{2]}\} + i\{p_1^{[3}p_2^{4]} - p_1^{[2}p_2^{1]}\} = 0 \\
&\{p_1^{[3}p_2^{1]} - p_1^{[4}p_2^{2]}\} - i\{p_1^{[3}p_2^{4]} - p_1^{[2}p_2^{1]}\} = 0
\end{aligned}\right\} \tag{2.10}$$

and

$$\left.\begin{aligned}
&\sum_{i=1}^{4} p_1^i p_1^i = 0 \qquad \sum_{i=1}^{4} p_2^i p_2^i = 0 \\
&\sum_{i=1}^{4} p_1^i p_2^i + 2i\{p_1^{[1}p_2^{4]} + p_1^{[3}p_2^{2]}\} = 0 \\
&\sum_{i=1}^{4} p_1^i p_2^i - 2i\{p_1^{[1}p_2^{4]} + p_1^{[3}p_2^{2]}\} = 0 \\
&\{p_1^{[1}p_2^{3]} - p_1^{[4}p_2^{2]}\} + i\{p_1^{[3}p_2^{4]} + p_1^{[2}p_2^{1]}\} = 0 \\
&\{p_1^{[1}p_2^{3]} - p_1^{[4}p_2^{2]}\} - i\{p_1^{[3}p_2^{4]} + p_1^{[2}p_2^{1]}\} = 0
\end{aligned}\right\}. \tag{2.11}$$

which can be reduced to the following three equations:

$$\sum_{i=1}^{4} p_a^i p_b^i = 0 \qquad (a, b = 1, 2) . \tag{2.12}$$

If after (2.3) we define the vector ρ_a^i $(a = 1, 2)$ by the equations:

$$\rho_a^i h_i^j = p_a^j , \tag{2.13}$$

we have, from (2.12),

$$g_{ij}\rho_a^i\rho_b^j = 0 \qquad (a, b = 1, 2). \tag{2.14}$$

In this case from (2.10) and (2.12) we have the solution of Ψ of the type either $\Psi_1 = \Psi_2 = 0$, or $\Psi_3 = \Psi_4 = 0$. For, otherwise from (2.10) and (2.11), we have the relation

$$\frac{p_1^1}{p_2^1} = \frac{p_1^2}{p_2^2} = \frac{p_1^3}{p_2^3} = \frac{p_1^4}{p_2^4},$$

which shows that p_a^i are not independent.

§ 3. The parallel-displacements which make $\rho^i\gamma_i\Psi = 0$ invariant.

In this section we will consider the parallel-displacements which make $\rho^i\gamma_i\Psi = 0$ invariant. Let two vectors $\rho^i(x)$ and $\bar{\rho}^i$ at any two consecutive points x^i and $x^i+\delta x^i$, be parallel to each other; then for the function set $(\gamma_i, \Psi, \rho^i)_x$ and $(\gamma_i, \Psi, \bar{\rho}^i)_{x+\delta x}$ at the two points, $\rho^i\gamma_i\Psi$ must be invariant. So we have the following relations:

$$\rho^i\gamma_i\Psi = 0, \tag{3.1}$$

$$\bar{\rho}^i(\gamma_i\Psi)_{x+\delta x} = 0, \tag{3.2}$$

$$\bar{\rho}^i = \rho^i - \Gamma_{jk}^i\rho^j\delta x^k, \tag{3.3}$$

where Γ_{jk}^i are functions of x's which are the general coefficients of connection in x-space. Expanding (3.2), substituting (3.3) und using (3.1), we have

$$\delta x^n\left\{\rho^i\frac{\partial}{\partial x^n}(\gamma_i\Psi) - \Gamma_{jn}^i\rho^j\gamma_i\Psi\right\} = 0.$$

This must hold for all δx^n: So we have

$$\rho^i\frac{\partial}{\partial x^n}(\gamma_i\Psi) - \Gamma_{jn}^i\rho^j\gamma_i\Psi = 0. \tag{3.4}$$

Since from (3.1) we have

$$\rho^i\frac{\partial}{\partial x^n}(\gamma_i\Psi) = -\frac{\partial\rho^i}{\partial x^n}\gamma_i\Psi,$$

(3.4) can be written in the form:

$$(\nabla_n \rho^i)\gamma_i \psi = 0 \tag{3.5}$$

where
$$\nabla_n \rho^i = \frac{\partial \rho^i}{\partial x^n} + \Gamma^i_{jn}\rho^j . \tag{3.6}$$

So we have the result: *The parallel-displacements which make $ds\psi = 0$ invariant are charactérised by the relations:*

$$(\nabla_n \rho^i)\gamma_i \psi = 0$$

and
$$\rho^i \gamma_i \psi = 0 .$$

Case (I) when the rank of matrix (2.6) is *3*. Since there is a vector ρ^i which satisfies (3.1), the condition that (3.1) and (3.5) are compatible is that the following equations:

$$g_{ij}\rho^i\rho^j = 0 \tag{3.7}$$

and
$$\nabla_n \rho^i = \sigma_n \rho^i \tag{3.8}$$

hold, where σ_n is any covariant vector.

Case (II) when the rank of the matrix (2.6) is *2*. Since there exist two independent vectors ρ^i_a $(a = 1, 2)$ which satisfy (3.1), the condition that (3.1) and (3.5) are compatible is equivalent to the following equations:

$$g_{ij}\rho^i_a\rho^j_b = 0 \tag{3.9}$$

and
$$\nabla_n \rho^i_a = \sum_{b=1}^{2} \sigma^b_{an}\rho^i_b \qquad (a, b = 1, 2) \tag{3.10}$$

hold, where σ^b_{an} are any covariant vectors with respect to suffices n.

§ 4. The condition for integrability of the equation for ψ.

Now let us consider the condition for integrability of

$$\rho^i\left(\frac{\partial \gamma_i \psi}{\partial x^m} - \Gamma^j_{im}\gamma_j \psi\right) = 0 \tag{4.1}$$

which is accompanied by the equation

$$\rho^i \gamma_i \psi = 0$$

where we regard the equation (4.1) as the differential equation for ψ when γ_i and Γ^i_{jm} are given.

Rewriting (4.1), we have

$$p^i \overset{\circ}{\gamma}_i \frac{\partial \overline{\psi}}{\partial x^m} = p^i A^l_{im} \overset{\circ}{\gamma}_l \overline{\psi} \tag{4.2}$$

where
$$\overline{\psi} = U^{-1}\psi$$

and
$$A^l_{im} = (-T_{km}^{\cdot\cdot j} + C^{hj}_m g_{hk}) \bar{h}^k_i h^l_j \text{ (1)}$$

To avoid confusions hereafter we write ψ instead of $\overline{\psi}$.

Now we shall consider the problem in two cases when the rank of the matrix (2.6) is *3* and *2*.

Case (I), when the rank of the matrix (2.6) is *3*. Solving p^i from (2.5) we have

$$\frac{p^1}{i(a-b)} = \frac{p^2}{1-ab} = \frac{p^3}{i(1+ab)} = \frac{p^4}{(a+b)} \tag{4.3}$$

where
$$a = \frac{\psi_3}{\psi_4}, \qquad b = \frac{\psi_1}{\psi_2}$$

Substituting (4.3) into (4.2) and writing in a actual form, we have

$$\left.\begin{aligned}
-2ib\frac{\partial a}{\partial x^m} &= p^i\{(A^1_{im} - A^4_{im}i)a + (-A^2_{im}i + A^3_{jm})\} \\
-2i\frac{\partial a}{\partial x^m} &= p^i\{-(A^2_{im}i + A^3_{im})a + (A^1_{im} + A^4_{im}i)\} \\
2ia\frac{\partial b}{\partial x^m} &= p^i\{(A^1_{im} + A^4_{im})b + (A^2_{im}i - A^3_{im})\} \\
2i\frac{\partial b}{\partial x^m} &= p^i\{(A^2_{im}i + A^3_{im})b + (A^1_{im} - A^4_{im}i)\}
\end{aligned}\right\} \tag{4.4}$$

Eliminating $\frac{\partial a}{\partial x^m}$ and $\frac{\partial b}{\partial x^m}$ from the above, we have the following equation:

(1) As seen in Morinaga's previous paper C^{ij}_m are the coefficient of $\gamma_{[i}\gamma_{j]}$ in the expansion Γ_m by sedenion and $T_{km}^{\cdot\cdot j} = \Gamma^j_{km} - \left\{ {j \atop km} \right\}$. See loc. cit. p. 160.

$$p^i\{(A^1_{im}-A^4_{im}i)a+(-A^2_{im}i+A^3_{im})+(A^2_{im}i+A^3_{im})ab-(A^1_{im}+A^4_{im}i)b\} = 0$$

Solving a and b from (4.3) in terms of p's and substituting them in the above, we have

$$\rho^i\rho^j T_{imj} = 0$$

or

$$\rho^i\rho^j \nabla_m g_{ij} = 0 \tag{4.5}$$

Differentiating (4.1) by x^l, and using the condition:

$$\frac{\partial^2 \gamma_i \Psi}{\partial x^{[l}\partial x^{m]}} = 0$$

we have

$$\frac{\partial \rho^i}{\partial x^{[l}}\left(\frac{\partial \gamma_i \Psi}{\partial x^{m]}} - \Gamma^j_{im]}\gamma_j\Psi\right) + \rho^i\left(-\frac{\partial \Gamma^j_{im]}}{\partial x^{[l}}\gamma_j\Psi - \Gamma^j_{i[m}\frac{\partial \gamma_j \Psi}{\partial x^{l]}}\right) = 0 \tag{4.6}$$

and using (3.5), the above condition can be written in the form:

$$\rho^i R_{lmi}{}^{\cdot\cdot\cdot j}\gamma_j\Psi = 0 \tag{4.7}$$

By successive differentiation of (4.5) and (4.7), we have, respectively,

$$\left.\begin{array}{l}\rho^i\rho^j T_{mij,k} = 0\\ \rho^i\rho^j T_{mij,ks} = 0\\ \cdots\cdots\cdots\cdots\end{array}\right\} \tag{4.8}$$

and

$$\left.\begin{array}{l}\rho^i R_{lmi,k}{}^{\cdot\cdot\cdot j}\gamma_j\Psi = 0\\ \rho^i R_{lmi,ks}{}^{\cdot\cdot\cdot j}\gamma_j\Psi = 0\\ \cdots\cdots\cdots\cdots\end{array}\right\} \tag{4.9}$$

where ρ^i are given by (2.3) and (4.3) in terms of Ψ.

So we have the result: *When the rank of the matrix* (2.6) *is 3, the necessary and sufficient condition for integrability of* (4.1) *is that the system of equations* (4.5), (4.7), (4.8) *and* (4.9) *is compatible.*

Further, we will investigate the above result in detail.

(A) Simplest case. First, we consider the case when (4.1) is completely integrable. In this case (4.5) and (4.7) must hold identically for all values of Ψ and accordingly for all values of a and b. So we have

$$\nabla_m g_{ij} = -g_{ij} Q_m \tag{4.10}$$

where Q_m is any covariant vector, i. e. the x-space is Weyl (not necessarily symmetric). And since, from (4.10), we have

$$R_{lm(ij)} = \frac{\partial Q_{l]}}{\partial x^{[m}} g_{ij},$$

the matrix equation (4.7) is reduced to the following two equations:

$$2a(k_{lm14}+k_{lm23})+(1-a^2)(k_{lm12}+k_{lm34})+i(1+a^2)(k_{lm13}-k_{lm24}) = 0, \tag{4.11}$$

$$2b(k_{lm14}-k_{lm23})+i(1+b^2)(k_{lm13}+k_{lm24})+(1-b^2)(k_{lm12}-k_{lm34}) = 0 \tag{4.12}$$

where
$$k_{lmij} = R_{lmp}^{\cdot\cdot\cdot q} \bar{h}^p_{[i} h^{j]}_q .$$

Since (4.11) and (4.12) must hold identically for all values of a and b, we have

$$R_{lmij} = -\frac{\partial Q_{m]}}{\partial x^{[l}} g_{ij} .^{(1)} \tag{4.13}$$

Hence we have the result: *When the rank of the matrix* (2.6) *is 3, the necessary and sufficient condition in order that* (4.1) *may be completely integrable, is that the x-space is Weyl* (*not necessarily symmetric*) *with the relation* (4.13).

General case. In the general case we have succeeded, but the result is so complicated that we do not feel much interest in it, so we confine ourselves here to the important case in which the x-space is Weyl (not necessarily symmetric).

In this case (4.5) and (4.8) are satisfied identically, therefore the condition for integrability of (4.1) is that the system of equations (4.7) and (4.9) namely (4.11), (4.12) and (4.9) is compatible for all suffices l and m. The condition that (4.11) has a common solution of a for the suffices l, m and l', m' is that

$$\begin{aligned}&\{(k_{lm12}+k_{lm34})(k_{l'm'13}-k_{l'm'24})-(k_{l'm'12}+k_{l'm'34})(k_{lm13}-k_{lm24})\}^2\\&+\{(k_{lm13}-k_{lm24})(k_{l'm'14}+k_{l'm'23})-(k_{l'm'13}-k_{l'm'24})(k_{lm14}+k_{lm23})\}^2\\&+\{(k_{lm14}+k_{lm23})(k_{l'm'12}+k_{l'm'34})-(k_{l'm'14}+k_{l'm'23})(k_{lm12}+k_{lm34})\}^2 = 0\end{aligned}$$

(1) These results are equivalent to those which were obtained in Morinaga's previous papers **6** (1935), 107.

which can be reduced to the following equations

$$\{\sum_p (k_{lm[2|p}k_{l'm'|3]p}-k_{lm[1|p}k_{l'm'|4]p})\}^2+\{\sum_p (k_{lm[1|p}k_{l'm'|2]p}-k_{lm[3|p}k_{l'm'4|p]})\}^2$$
$$+\{\sum_p (k_{lm[1|p}k_{l'm'|3]p}+k_{lm[2|p}k_{l'm'|4]p})\}^2 = 0\,.$$

By the same method which was used in the previous paper,(1) the above equations become

$$\epsilon^{ijkr}R_{lmip}R_{l'm'j}^{\cdot\ \cdot\ \cdot\ p}R_{lmkq}R_{l'm'r}^{\cdot\ \cdot\ \cdot\ q} = 2\sqrt{\Delta}\,R_{lmip}R_{l'm'j}^{\cdot\ \cdot\ \cdot\ p}R_{lm\cdot q}^{\cdot\ \cdot\ i}R_{l'm'}^{\cdot\ \cdot\ jq}\,. \tag{4.14}$$

Similarly, from (4.12) we have

$$\epsilon^{ijkr}R_{lmip}R_{l'm'j}^{\cdot\ \cdot\ \cdot\ p}R_{lmkq}R_{l'm'r}^{\cdot\ \cdot\ \cdot\ q} = -2\sqrt{\Delta}\,R_{lmip}R_{l'm'j}^{\cdot\ \cdot\ \cdot\ p}R_{lm\cdot q}^{\cdot\ \cdot\ i}R_{l'm'}^{\cdot\ \cdot\ jq}\,. \tag{4.15}$$

Hence the general condition of integrability of (4.1) when the x-space is Weyl (not necessarily symmetric), is that

$$\epsilon^{ijkr}R_{lmip}R_{l'm'j}^{\cdot\ \cdot\ \cdot\ p}R_{lmkq}R_{l'm'r}^{\cdot\ \cdot\ \cdot\ q} = 0\,,$$

and

$$R_{lmip}R_{l'm'j}^{\cdot\ \cdot\ \cdot\ p}R_{lm\cdot q}^{\cdot\ \cdot\ i}R_{l'm'}^{\cdot\ \cdot\ jq} = 0\,.$$

Case (II), when the rank of the matrix (2.6) is 2. From the statements in § 2 it must be that $\psi_1=\psi_2=0$ or $\psi_3=\psi_4=0$. First, we will consider the case when $\psi_3=\psi_4=0$. In this case we have, from (2.5),

$$\left.\begin{aligned} &\frac{p_1^1}{2ib} = \frac{p_1^2}{-(b^2+1)} = \frac{p_1^3}{i(b^2-1)} = \frac{p_1^4}{0} \\ &\frac{p_2^1}{b^2+1} = \frac{p_2^2}{2ib} = \frac{p_2^3}{0} = \frac{p_2^4}{i(b^2-1)}\,, \end{aligned}\right\} \tag{4.16}$$

Substituting these values of $p_a^i\,(a=1,2)$ into (4.2), and eliminating from the resulting equations, we have

$$p_1^i\{(A_{im}^1+A_{im}^4 i)b+(iA_{im}^2-A_{im}^3)+(iA_{im}^2+A_{im}^3)b^2+(A_{im}^1-A_{im}^4 i)b\} = 0\,,$$

$$p_2^i\{(A_{im}^1+A_{im}^4 i)b^2+(iA_{im}^2-A_{im}^3)b+(iA_{im}^2+A_{im}^3)b+(A_{im}^1-iA_{im}^4)\} = 0\,,$$

and

$$p_1^i\{(A_{im}^1+iA_{im}^4)b^2+(iA_{im}^2-A_{im}^3)b+(iA_{im}^2+A_{im}^3)b+(A_{im}^1-iA_{im}^4)\}$$
$$+p_2^i\{(A_{im}^1+iA_{im}^4)b+(iA_{im}^2-A_{im}^3)+(iA_{im}^2+A_{im}^3)b^2+(A_{im}^1-iA_{im}^4)b\} = 0$$

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(1) K. Morinaga, this journal **5** (1935), 173.

and by aid of (4.16) the above can be reduced to the simple equation:

$$p^i_a p^j_b A^j_{im} = 0$$

or

$$\rho^i_a \rho^j_b \nabla_m g_{ij} = 0\,. \tag{4.17}$$

where ρ^i_a are given by (2.3) and (4.16).

By the same method, by which (4.7), (4.8) and (4.9) were obtained, from (4.1) we have the following equations:

$$\rho^i_a R_{lmi}^{\cdot\cdot\cdot j} \gamma_j \Psi = \,, \tag{4.18}$$

$$\left.\begin{array}{l} \rho^i_a \rho^j_b T_{mij,k} = 0 \\ \cdots\cdots\cdots\cdots \end{array}\right\}, \tag{4.19}$$

$$\left.\begin{array}{l} \rho^i_a R_{lmi,k}^{\cdot\cdot\cdot j} \gamma_j \Psi = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \end{array}\right\}. \tag{4.20}$$

Hence we have the result: *When the rank of the matrix* (2.6) *is 2, the condition for integrability of* (4.1) *is that the system of equations* (4.17), (4.18), (4.19) *and* (4.20) *is compatible.*

By the same method which was used in the case where the rank of the matrix (2.6) is *3*, as the condition in order that (4.1) may be completely integrable, we can easily get from (4.1) the following equations:

$$\nabla_m g_{ij} = -g_{ij} Q_m \tag{4.21}$$

and

$$k_{lm12} = k_{lm34}\,, \qquad k_{lm13} = -k_{lm24}\,, \qquad k_{lm14} = k_{lm23}\,.$$

the later is rewritten as[1]

$$\frac{\pm\sqrt{\Delta}}{2} \epsilon_{ijpq} R_{lm}^{\cdot\cdot pq} = R_{lmij} \tag{4.22}$$

Similarly, when $\Psi_1 = \Psi_2 = 0$, corresponding to (4.21), we have

(1) loc. cit. 171.

$$\frac{\mp\sqrt{\varDelta}}{2}\epsilon_{ijpq}R_{lm}^{\cdot\cdot\,pq} = -R_{lmij} \text{ (1)} \tag{4.23}$$

So we have the result: *When the rank of the matrix* (2.6) *is 2, the necessary and sufficient condition in order that* (4.1) *may be completely integrable, is that* (4.21) *and* (4.22) *or* (4.21) *and* (4.23) *are compatible.*

By the same method as in the general case of (1), we get the result: *When the space is Weyl* (*not necessarily symmetric*), *the general condition for integrability of* (4.1) *is that*

$$\pm\epsilon^{ijkr}R_{lmip}R_{l'm'j}^{\cdot\ \cdot\ \cdot\ p}R_{lmkq}R_{l'm'r}^{\cdot\ \cdot\ \cdot\ q} = 2\sqrt{\varDelta}\,R_{lmip}R_{l'm'j}^{\cdot\ \cdot\ \cdot\ p}R_{lm\cdot q}^{\cdot\ \cdot\ i}R_{l'm'}^{\cdot\ \cdot\ jq}$$

or

$$\pm\epsilon^{ijkr}R_{lmip}R_{l'm'j}^{\cdot\ \cdot\ \cdot\ p}R_{lmkq}R_{l'm'r}^{\cdot\ \cdot\ \cdot\ q} = -2\sqrt{\varDelta}\,R_{lmip}R_{l'm'j}^{\cdot\ \cdot\ \cdot\ p}R_{lm\cdot q}^{\cdot\ \cdot\ i}R_{l'm'}^{\cdot\ \cdot\ jq}\,.$$

§ 5. Fundamental Equation for ψ.

In this section, we shall see how the equation (4.1) is related to the fundamental equation for ψ:

$$\frac{\partial\psi}{\partial x^m} = (\Gamma_m + T^{i\cdot j}_{\cdot m}\gamma_i\gamma_j + T_m\gamma_5)\psi$$

obtained in previous paper.[2]

From the equation (4.1), we have

$$\frac{\partial\gamma_i\psi}{\partial x^m} = (\Gamma^j_{im}\gamma_j + \Lambda_m\gamma_i)\psi \tag{5.1}$$

where Λ_m is a certain 4–4 matrix. Now we put the assumption (a): the rank of the matrix $\gamma_i\psi$ is 2, the equation (5.1) is completely integrable for ψ, and Λ_m is independent of the initial values of ψ.

After a suitable S-transformation, (5.1) can be brought to the form:

(1) Double signs of (4.22) and (4.23) are to be taken correspondingly in order that (4.22) and (4.23) must be equivalent, because the two cases of (4.22) and (4.23) are interchangeable by a suitable S-transformation without changing the coefficients of connection and the metrics g_{ij}'s.

(2) loc. cit., 162–165.

$$\mathring{\gamma}_i \frac{\partial \overline{\Psi}}{\partial x^m} = \{\overline{\Gamma}^j_{im}\mathring{\gamma}_j + \overline{\Lambda}_m \mathring{\gamma}_i + 4\overline{C}^{ji}_m \mathring{\gamma}_j\}\overline{\Psi} \tag{5.2}$$

where

$$\overline{\Gamma}^j_{im} = \Gamma^p_{qm} h^j_p \bar{h}^q_i, \qquad \overline{C}^{ji}_m \gamma_j \gamma_i = \overline{\Gamma}_m \qquad \overline{\Lambda}_m = \Lambda_m - S\frac{\partial \overline{S}}{\partial x^m}$$

But since $\mathring{\gamma}_i$ have the form:

$$\mathring{\gamma}_i = \begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & 0 & 0 \\ \times & \times & 0 & 0 \end{pmatrix}$$

and Λ_m are independent of every initial values of ψ, from (5.2) we can conclude that $\overline{\Lambda}_m$ are expanded only in the terms $\mathring{\gamma}_{[i}\mathring{\gamma}_{j]}$ $(i, j = 1, \ldots, 4)$ and do not involve the terms $\mathring{\gamma}_5\mathring{\gamma}_i$ $(i = 1, \ldots, 4)$ and $\mathring{\gamma}_i$ $(i = 1, \ldots, 4)$. Hence multiplying (5.2) by $\mathring{\gamma}_i$ and contracting by i, we have

$$\frac{\partial \psi}{\partial x^m} = (\Gamma_m + T_{im}^{\cdot\cdot j}\gamma_i\gamma_j + T_m\gamma_5)\psi \tag{5.3}$$

which is identical with the result obtained in former paper.[1]

Specially, when the x-space is Riemannian, (5.3) becomes as follows:

$$\frac{\partial \psi}{\partial x^m} = (\Gamma_m + T_m\gamma_5)\psi\,. \tag{5.4}$$

So we have the result[1] from the equation (6.9) in previous paper: *Under the assumption* (a), *in a Riemannian space whose parallelism is admitted by* $ds\psi = 0$, *the fundamental equation for* ψ *is*

$$\frac{\partial \psi}{\partial x^m} = (\Gamma_m + T_m\gamma_5)\psi$$

and the condition of comylete integrability of the above equation is that

$$\frac{\pm\sqrt{\Delta}}{2}\epsilon_{ijpq}K_{lm}^{\cdot\cdot pq} = K_{lmij}\,.$$

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(1) loc. cit., 167.

§ 6. The metrics.

Here we shall consider only the case in which the rank of $(\gamma_i\psi)$ is 2.[1]

(1) When the x-space is Weyl (symmetric) we will obtain the general solution of g_{ij} of the wave geometry which makes $ds\psi = 0$ invariant. For this purpose we consider the equations (3.9) and (3.10):

$$g_{ij}\rho^i_a\rho^j_b = 0 \tag{6.1}$$

$$\frac{\partial \rho^i_a}{\partial x^n} + \Gamma^i_{ln}\rho^l_a = \sigma^c_{an}\rho^i_c \qquad (a, b = 1, 2) \tag{6.2}$$

multiplying (6.2) by ρ^n_b $(b \neq a)$, contracting it by n, and substracting from this equation the equation obtained by interchanging a and b, we have

$$\left(\rho^n_b \frac{\partial \rho^i_a}{\partial x^n} - \rho^n_a \frac{\partial \rho^i_b}{\partial x^n}\right) = (\sigma^c_{an}\rho^n_b - \sigma^c_{bn}\rho^n_a)\rho^i_c. \quad (a, b, c = 1, 2) \tag{6.3}$$

This shows that the system of differential equations:

$$\rho^i_a \frac{\partial f}{\partial x^i} = 0 \qquad (a = 1, 2) \tag{6.4}$$

makes a complete system. Therefore (6.4) has 2 independent solutions, say $u^3(x^1, \ldots, x^4)$ and $u^4(x^1, \ldots, x^4)$. If we change the variables by

$$\bar{x}^1 = x^1, \quad \bar{x}^2 = x^2, \quad \bar{x}^3 = u^3(x^1, \ldots, x^4), \quad \bar{x}^4 = u^4(x^1, \ldots, x^4) \tag{6.5}$$

and take suitable linear combination of ρ^i_a, we can put

$$\rho^i_a = \delta^i_a \qquad (a = 1, 2). \tag{6.6}$$

In this coordinate-system, (6.1) and (6.2) become, respectively

$$g_{ab} = 0 \tag{6.7}$$

$$\Gamma^i_{an} = \sigma^c_{an}\delta^i_c \qquad (a, b, c = 1, 2). \tag{6.8}$$

From (6.8) we have

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(1) Because the case where the rank is 3 becomes trivial when the x-space becomes Riemannian, and it is not looked upon as of much importance in physical applications.

$$\Gamma^{\lambda}_{an} = 0 \qquad (a = 1, 2\,;\ \lambda = 3, 4)\,. \tag{6.9}$$

(Hereafter we assume that $a, b, c = 1, 2$; $i, j, k = 1, \ldots, 4$ and $\lambda, \mu = 3, 4$).
From (6.9), we have

$$g^{\lambda i}\left\{\frac{\partial g_{ia}}{\partial x^n} + \frac{\partial g_{in}}{\partial x^a} - \frac{\partial g_{an}}{\partial x^i}\right\} + \delta^{\lambda}_{a} Q_n + \delta^{\lambda}_{n} Q_a - Q_i g^{i\lambda} g_{an} = 0\,. \tag{6.10}$$

Since $g_{ab} = 0$ and the determinant $|g_{ij}|$ must not vanish identically, it must be that

$$g^{\lambda\mu} = 0 \qquad \text{and} \qquad |g_{a\lambda}| \neq 0\,.$$

Hence from (6.10), we have

$$\left(\frac{\partial g_{b\lambda}}{\partial x^a} - \frac{\partial g_{a\lambda}}{\partial x^b}\right) + Q_a g_{b\lambda} - Q_b g_{a\lambda} = 0 \tag{6.11}$$

or

$$g^{\lambda b}\left(\frac{\partial g_{\lambda b}}{\partial x^a} - \frac{\partial g_{\lambda a}}{\partial x^b}\right) = -Q_a\,. \tag{6.12}$$

When the x-space is Riemannian, from (6.11) we know that

$$g_{a\lambda} = \frac{\partial f_\lambda}{\partial x^a}\,.$$

where f_λ $(\lambda = 3, 4)$ are any functions of x's.

So we have the result: *When the rank of the matrix $\gamma_i\psi$ is 2 in a Riemannian space in which $ds\psi = 0$ is invariant, in a coordinate-system suitably chosen the metrics g_{ij} is obtained in the form*

$$\begin{cases} g_{ab} = 0\,, \qquad g_{\lambda\mu} = \text{arbitrary}, \qquad (a, b = 1, 2) \\ g_{a\lambda} = \dfrac{\partial f_\lambda}{\partial x^a} = g_{\lambda a} \qquad (\lambda = 3, 4) \end{cases}$$

where f_λ are arbitrary functions of x's.

§ 7. The general solution g_{ij} of the equation:

$$\frac{\pm\sqrt{\Delta}}{2}\,\epsilon_{ijpq} R_{lm}^{\cdot\,\cdot\,pq} = R_{lm[ij]}\,.$$

If we obtain the solution g_{ij} when (4.1) is completely integrable and the rank of $(\gamma_i\psi)$ is 2, from the result obtained in § 4, such g_{ij} must be the general solution of the equation:

$$\frac{\pm\sqrt{\Delta}}{2}\epsilon_{ijpq}R_{lm}^{\cdot\cdot pq} = R_{tm[ij]}\,. \tag{7.1}$$

Since (4.1) is completely integrable, there may be at least two independent solutions ψ, say $\bar{\psi}$ and $\bar{\bar{\psi}}$, which are not related by $\bar{\psi} = \mu\bar{\bar{\psi}}$. And since the rank of $(\gamma_i\psi)$ is 2, there exist two vectors ρ^i for each ψ, say $\bar{\rho}^i_a\,(a = 1, 2)$ for $\bar{\psi}$, $\bar{\bar{\rho}}^i_a\,(a = 1, 2)$ for $\bar{\bar{\psi}}$; which satisfy the relations:

$$\bar{\rho}^i_a\gamma_i\bar{\psi} = 0 \tag{7.2}$$

and

$$\bar{\bar{\rho}}^i_a\gamma_i\bar{\bar{\psi}} = 0\,. \tag{7.3}$$

It is proved that the four vectors $\bar{\rho}^i_a$, $\bar{\bar{\rho}}^i_a$ are independent of each other. For, if one of them, say $\bar{\bar{\rho}}^i_2$, is expressed linearly by the others, say $\bar{\bar{\rho}}^i_2 = A\bar{\rho}^i_1+B\bar{\rho}^i_2+C\bar{\bar{\rho}}^i_1$, it must be that from (7.3)

$$(A\bar{\rho}^i_1+B\bar{\rho}^i_2)\gamma_i\bar{\bar{\psi}} = 0$$

i. e.

$$\bar{\psi} = \mu\bar{\bar{\psi}}$$

which is a contradiction

Substituting $\bar{\rho}^i_a$ and $\bar{\bar{\rho}}^i_a$ into (6.1) and (6.2), and writing ρ^i_a for $\bar{\rho}^i_a$ and $\rho^i_\lambda\,(\lambda = 3, 4)$ for $\bar{\bar{\rho}}^i_a\,(a = 1, 2)$, similarly as in § 6, it is proved that each of the two systems of differential equations:

$$\rho^i_a\frac{\partial f}{\partial x^i} = 0 \qquad (a = 1, 2) \tag{7.4}$$

and

$$\rho^i_\lambda\frac{\partial f}{\partial x^i} = 0 \qquad (\lambda = 3, 4) \tag{7.5}$$

make a complete systems. Therefore (7.4) and (7.5) have each two independent solutions, say $u^3(x)$, $u^4(x)$ and $u^1(x)$, $u^2(x)$, respectively. Then by the coordinate-transformation:

$$\bar{x}^i = u^i(x) \qquad (i = 1, \ldots\ldots, 4)$$

and by linear combination of ρ^i_j's, we can put

$$\rho^i_j = \delta^i_j$$

So from (6.1) and (6.2), we have

$$g_{ab} = 0\,, \qquad g_{\lambda\mu} = 0\,, \tag{7.6}$$

$$\Gamma^{\lambda}_{an} = 0\,, \qquad \Gamma^{a}_{\lambda n} = 0 \tag{7.7}$$

Therefore, from (7.7)

$$g^{\lambda b}\left(\frac{\partial g_{b\lambda}}{\partial x^a} - \frac{\partial g_{a\lambda}}{\partial x^b}\right) = -Q_a\,, \tag{7.8}$$

$$g^{\mu b}\left(\frac{\partial g_{\mu b}}{\partial x^\lambda} - \frac{\partial g_{\lambda b}}{\partial x^\mu}\right) = -Q_\lambda\,. \tag{7.9}$$

Substituting (7.6), (7.8) and (7.9) into (7.1), we have

$$\left.\begin{aligned} R_{c\mu 1}^{\cdot\cdot\cdot 1} + R_{c\mu 2}^{\cdot\cdot\cdot 2} &= \frac{\partial Q_c}{\partial x^\mu} - \frac{\partial Q_\mu}{\partial x^c} \\ R_{c\mu 3}^{\cdot\cdot\cdot 3} + R_{c\mu 4}^{\cdot\cdot\cdot 4} &= \frac{\partial Q_c}{\partial x^\mu} - \frac{\partial Q_\mu}{\partial x^c} \end{aligned}\right\} . \tag{7.10}$$

On the other hand, we have from the expression of $R_{c\mu i}^{\cdot\cdot\cdot i}$ actually

$$R_{c\mu i}^{\cdot\cdot\cdot i} = \frac{\partial}{\partial x^\mu}\left\{g^{i\lambda}\frac{\partial g_{i\lambda}}{\partial x^c} + Q_c\right\}. \qquad (i:\ \text{not sum})$$

Substituting the above into (7.10), we have

$$\frac{\partial^2}{\partial x^\mu \partial x^c}(\log\sigma) + \frac{\partial Q_c}{\partial x^\mu} + \frac{\partial Q_\mu}{\partial x^c} = 0 \tag{7.11}$$

where

$$\sigma = \begin{vmatrix} g_{13} & g_{14} \\ g_{23} & g_{24} \end{vmatrix}.$$

From the condition of the integrability of (7.11) for $\log\sigma$, we have

$$\frac{\partial Q_c}{\partial x^\mu} = \frac{\partial Q_\mu}{\partial x^c} + K_{c\mu}$$

where $K_{c\mu}$ are any constants.

So we have the result: *When the x-space is Weyl in which* $ds\psi = 0$ *is invariant, the general solution* g_{ij} *and* Q_i *of the equation* (7.1) *are given by* (7.6), (7.8), (7.9) *and* (7.11) *in a coordinate-system suitably chosen.*

Specially when $Q_i = 0$, we have the result: *When the x-space is*

Riemannian, in a coordinate system suitably chosen the general solution of the equation (7.1) *is given by*

$$g_{ab} = 0, \qquad g_{a\lambda} = \frac{\partial^2 f}{\partial x^a \partial x^\lambda}, \qquad g_{\lambda\mu} = 0$$

where f is any function satisfying the relation :

$$\frac{\partial^2 f}{\partial x^1 \partial x^3} \frac{\partial^2 f}{\partial x^2 \partial x^4} - \frac{\partial^2 f}{\partial x^2 \partial x^3} \frac{\partial^2 f}{\partial x^2 \partial x^4} = \varphi(x^1, x^2) \chi(x^3, x^4)$$

φ and χ being arbitrary functions of their arguments.

This problem was discussed at a special Seminar of Geometry and Theoretical Physics of this University.

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