Projective Wave Geometry.

By

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§ 1. Introduction.

Mr. K. Morinaga has constructed the wave geometry[1] in which

$$ds\psi \equiv \gamma_i dx^i \psi = 0^{(2)} \tag{1.1}$$

is invariant. According to Morinaga, the fundamental equation for ψ is given by

$$\nabla_m \psi = R_m I \psi \qquad \text{where} \qquad \nabla_m \equiv \frac{\partial}{\partial x^m} + \Lambda_m - 2T^{\lambda}_{m}\gamma_{\lambda}\,. \tag{1.2}$$

And as the condition for integrability of this equation, the equation

$$\frac{\sqrt{g}}{2}\varepsilon_{stpq} K_{lm}^{\cdot\cdot pq} = \pm K_{lmst}\,, \tag{1.3}$$

where $$g = \text{det.}\,|\,g_{ij}\,|\,,$$

was introduced.[3]

The purpose of this paper is to construct a geometry by applying the principle of linearisation[4] to O. Veblen's projective relativity[5] and by using the method of Morinaga's wave geometry.

§ 2. Principle of linearisation.

In order to apply the principle of linearisation to projective relativity, we must first express Veblen's metric in terms of wave geometry.

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(1) K. Morinaga, This Journal, **5** (1935), 151.

(2) Throughout this paper we assume that the Latin suffices vary 1, 2, 3, 4 and the Greek suffices 0, 1, 2, 3, 4.

(3) K. Morinaga, ibid., 160 and 169.

(4) Y. Mimura, This Journal, **5** (1935), 99.

(5) O. Veblen, *Projektive Relativitätstheorie*, Berlin (1933).

In projective relativity the metric properties of space are characterized by a symmetric projective tensor $G_{\alpha\beta}$ of index $2N$. If we put

$$G_{00} = \Phi^2,$$

$$\frac{G_{\alpha\beta}}{G_{00}} = \gamma_{\alpha\beta}, \tag{2.1}$$

and

$$\frac{G_{0\alpha}}{G_{00}} = \varphi_\alpha,$$

then Φ is a projective scalar of index N, and both $\gamma_{\alpha\beta}$ and φ_α are the components of projective tensors of index 0, and are therefore functions of (x^1, x^2, x^3, x^4). If we put

$$\gamma_{\alpha\beta} - \varphi_\alpha\varphi_\beta = g_{\alpha\beta}, \quad \text{(consequently } g_{\alpha 0} = 0)$$

g_{ij} are the components of an affine tensor, which, we assume, determines the metric of our affine space. Thus in tangential space at every point of the base space a quadric

$$\gamma_{\alpha\beta}X^\alpha X^\beta = 0 \tag{2.2}$$

is determined, whose radius is equal to 1 and whose centre is at the tangential point. Applying the principle of linearisation to (2.2), we have

$$(\gamma_\alpha X^\alpha)(\gamma_\beta X^\beta) = 0,$$

where γ_α are five 4–4 matrices satisfying the equation

$$\gamma_{(\alpha}\gamma_{\beta)} = \gamma_{\alpha\beta} I.$$

Pauli has shown that the most general form of such γ_α is given by

$$\gamma_\alpha = U h_\alpha^\varepsilon \mathring{\gamma}_\varepsilon U^{-1}, \text{[1]} \tag{2.3}$$

where U is an arbitrary 4–4 matrix whose elements are supposed to be any functions of $(x^0, x^1, \ldots, x^4)$, $\mathring{\gamma}_\varepsilon$ are Dirac matrices,[2] and h_λ^μ are certain functions of the x's. These h_λ^μ are obtained as follows[3]:

(1) W. Pauli, Ann. der Physik, **18** (1933), 344.
(2) T. Hosokawa, This Journal, **5** (1935), 142. ($\mathring{\gamma}_i = E_i$ and $\mathring{\gamma}_0 = E_5$)
(3) T. Hosakawa, ibid., 145.

$$(h^{\mu}_{\lambda}) = \begin{pmatrix} 1 & \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 \\ 0 & & & & \\ 0 & & p^i_j & & \\ 0 & & & & \\ 0 & & & & \end{pmatrix} \quad \text{and} \quad (\bar{h}^{\mu}_{\lambda}) = \begin{pmatrix} 1 & \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ 0 & & & & \\ 0 & & q^i_j & & \\ 0 & & & & \\ 0 & & & & \end{pmatrix}, \tag{2.4}$$

where p^i_j, q^i_j and ϕ_i are the functions of (x^1, x^2, x^3, x^4) satisfying

$$p^i_j q^j_k = \delta^i_k, \qquad p^j_i q^k_j = \delta^k_i,$$

$$\left.\begin{aligned} g_{ij} &= a_{(i}a_{j)} \quad \text{where} \quad a_i = p^h_i \overset{\circ}{\gamma}_h, \quad \overset{\circ}{\gamma}_h = \overset{\circ}{\gamma}^h \\ g^{ij} &= a^{(i}a^{j)} \quad \text{where} \quad a^i = q^i_h \overset{\circ}{\gamma}^h, \end{aligned}\right\}$$

and
$$\phi_i = -q^k_i \varphi_k .$$

Then we have
$$h^{\mu}_{\lambda}\bar{h}^{\lambda}_{\nu} = \delta^{\mu}_{\nu}, \qquad h^{\mu}_{\lambda}\bar{h}^{\nu}_{\mu} = \delta^{\nu}_{\lambda},$$

and
$$a_i = \gamma_i - \varphi_i \gamma_0, \qquad a^i = \gamma^i . \tag{2.5}$$

And if we put
$$\gamma^{\alpha} = U\bar{h}^{\alpha}_{\beta}\overset{\circ}{\gamma}^{\beta}U^{-1} \qquad (\overset{\circ}{\gamma}^{\beta} = \overset{\circ}{\gamma}_{\beta})$$

we have
$$\gamma^{\alpha\beta} = \gamma^{(\alpha}\gamma^{\beta)} .$$

Veblen[1] used the spinors of the form:

$$\psi = e^{Mx^0}\psi'(x^1, x^2, x^3, x^4) . \tag{2.6}$$

We shall remove the above restriction and consider the spinors in general form in our geometry. The law of transformation of ψ will be given in the next section.

§ 3. Coordinate-transformations and spinor-transformations.

Now we assume that in *projective wave geometry* the transformations of coordinates and spinors occur in the form

$$\left.\begin{aligned} \bar{x}^i &= \bar{x}^i(x^1, x^2, x^3, x^4), \text{— coordinate-transformation in the affine space} \\ \bar{x}^0 &= x^0 + \log \rho(x^1, x^2, x^3, x^4), \text{— gauge transformation of } x^0 \end{aligned}\right\} \tag{3.1}$$

(1) O. Veblen, Proc. Nat. Acad. of Science. U.S.A. **19** (1933), 979.

$$\overline{\psi} = S\psi, \text{— spinor-transformation} \tag{3.2}$$

where S is a 4–4 matrix whose elements are arbitrary functions of $(x^0, x^1, \ldots, x^4)$. If we take the spinors of the form (2.6) and put the condition that the *index M* should remain constant under all the spinor-transformations, S must be a function of (x^1, x^2, x^3, x^4) only.

Further, we assume that *all the transformations (3.1) and (3.2) make*

$$\gamma_a X^a \psi = 0 \tag{3.3}$$

invariant.

From these assumptions we know, in the first place, that by the transformations (3.1) and (3.2), we have

$$\overline{\gamma_a X^a \psi} = \Omega \gamma_a X^a \psi , \tag{3.4}$$

where Ω is a 4–4 matrix. Then from (3.1), (3.2) and (3.4), we have

$$W\gamma_a W = \gamma_a , \qquad W = S^{-1}\Omega .\text{[1]} \tag{3.5}$$

In order to solve W from (3.5) we consider, for a while, the case in which the transformations (3.1) and (3.2) are infinitesimal, putting

$$W = I + \omega , \qquad \omega = t^{\alpha\beta}\gamma_\alpha\gamma_\beta + t^\alpha\gamma_\alpha + tI , \qquad t^{(\alpha\beta)} = 0 .$$

Then from (3.5) we have

$$\gamma_\lambda t^{\alpha\beta}\gamma_\alpha\gamma_\beta + 2t^{\alpha\beta}\gamma_\alpha\gamma_{\beta\lambda} + t^\alpha\gamma_{\lambda\alpha} + t\gamma_\lambda = 0 .$$

Multiplying this by γ^λ and summing for λ, we have

$$3t^{\alpha\beta}\gamma_\alpha\gamma_\beta + t^\alpha\gamma_\alpha + 5t = 0 .$$

Comparing the coefficients of each base of sedenion in the above, we have

$$\omega = 0 , \qquad W = I .$$

Then we finally obtain $\qquad \Omega = S . \qquad (3.6)$

So we have the result: *The most general transformation which makes* $\gamma_a X^a \psi = 0$ *invariant is given by*

(1) Analogously to the process in Morinaga's paper, ibid., 153.

$$\left.\begin{aligned} &\bar{x}^i = \bar{x}^i(x^1, \dots, x^4), \qquad \bar{x}^0 = x^0 + \log \rho(x^1, \dots, x^4), \\ &\bar{\Psi} = S\Psi, \\ &\bar{\gamma}_a = \frac{\partial x^\mu}{\partial \bar{x}^a} S\gamma_\mu S^{-1}. \end{aligned}\right\} \tag{3.7}$$

Therefore we know that, in our projective wave geometry, the transformation composed of C-transformation and S-transformation is the most general. Hence in our case the gauge transformation of γ's does not appear, though it does so in Morinaga's wave geometry, but the gauge transformation of x^0 appears instead. Thus we have obtained the same transformations as in ordinary spinor-calculus.(1)

Specially, we easily see from (3.6) that the transformation which makes $\gamma_a X^a \Psi$ itself invariant is given by

$$\Omega = S = I;$$

therefore the transformation is nothing but the C-transformation.

Next, we shall introduce the Lorentz transformation as a special case. If we assume that $\mathring{\gamma}_a = \bar{\gamma}_a = \gamma_a$ and (3.7) is infinitesimal, we have

$$(\delta^\mu_\lambda + u^\mu_\lambda)\mathring{\gamma}_\mu = (I+\sigma)\mathring{\gamma}_\lambda(I-\sigma) \tag{3.8}$$

where
$$S = I+\sigma, \qquad \frac{\partial \bar{x}^\mu}{\partial x^\lambda} = \delta^\mu_\lambda + u^\mu_\lambda.$$

Expanding σ in sedenion and calculating (3.8), we have

$$\sigma = \frac{1}{4} u^\alpha_\beta \mathring{\gamma}_\alpha \mathring{\gamma}^\beta = sI \qquad \text{and } u^\alpha_\beta = -u^\beta_\alpha. \tag{3.9}$$

But on the other hand, from (3.1), we have

$$\frac{\partial \bar{x}^\lambda}{\partial x^0} = \delta^\lambda_0 \qquad \text{and} \qquad u^\lambda_0 = 0;$$

therefore from (3.9),

$$u^\lambda_0 = u^0_\lambda = 0, \qquad u^i_j = -u^j_i, \tag{3.10}$$

and hence
$$\sigma = \frac{1}{4} u^i_j \mathring{\gamma}_i \mathring{\gamma}^j + sI. \tag{3.11}$$

(1) W. Pauli, loc. cit., 348.

Consequently our coordinate-transformation (3.1) becomes an ordinary affine transformation, and the gauge transformation does not appear. And from (3.11) we know that S is separated into two linear transformations

$$(\psi_1, \psi_2) \rightarrow (\overline{\psi}_1, \overline{\psi}_2)\,, \qquad (\psi_3, \psi_4) \rightarrow (\overline{\psi}_3, \overline{\psi}_4)\,.$$

Moreover from (3.10), we see that the affine coordinate-transformation is a rotation. Therefore we get the same result for the Lorentz transformation as in Morinaga's wave geometry.(1)

§ 4. The parallel displacements of $\gamma_a X^a \psi$.

We shall consider *the parallel displacements which make* $\gamma_a X^a \psi = 0$ *invariant*. We denote the coefficients of connection in projective space by $\Gamma^\tau_{\alpha\beta}$, which may be any functions of the x's. Then we have the relation

$$\left.\begin{aligned} (\gamma_a \overline{X}^a \psi)_x &= (I + \Lambda_\mu dx^\mu)(\gamma_a X^a \psi)_{x+dx}\,, \\ \overline{X}^\nu &= X^\nu + \Gamma^\nu_{\alpha\beta} X^\alpha dx^\beta\,, \end{aligned}\right\} \tag{4.1}$$

where Λ's are certain 4-4 matrices.

Using the identity(2)

$$\frac{\partial \gamma_\lambda}{\partial x^\mu} = \{^{\nu}_{\lambda\mu}\}\gamma_\nu + \Gamma_\mu \gamma_\lambda - \gamma_\lambda \Gamma_\mu\,, \tag{4.2}$$

where $\{^{\nu}_{\lambda\mu}\}$ is the Christoffel symbol made from $\gamma_{\alpha\beta}$, and Γ_μ are 4-4 matrices, having the following expansion:

$$\Gamma_\mu = C^{\alpha\beta}_\mu \gamma_\alpha \gamma_\beta + C^\alpha_\mu \gamma_\alpha\,, \qquad C^{(\alpha\beta)}_\mu = 0\,,$$

we have the following relations after the same calculations as Morinaga made(3):

$$\Gamma^\tau_{\mu\nu} = \{^{\,\tau}_{\mu\nu}\} + 4(L^{\tau\beta}_\nu + C^{\tau\beta}_\nu)\gamma_{\beta\mu} + \delta^\tau_\mu R_\nu\,, \tag{4.3}$$

$$L^\beta_\nu + C^\beta_\nu = 0\,, \tag{4.4}$$

(1) K. Morinaga, loc. cit., 158.
(2) W. Pauli, ibid., 356.
(3) K. Morinaga, ibid., 160.

where R_ν is an arbitrary covariant vector, and $L_\nu^{\alpha\beta}$ and L_ν^β are the coefficients of the expansion of Λ_ν:

$$\Lambda_\nu = L_\nu^{\alpha\beta}\gamma_\alpha\gamma_\beta + L_\nu^\alpha\gamma_\alpha + L_\nu I\,, \qquad L_\nu^{(\alpha\beta)} = 0\,.$$

Therefore our case corresponds to the special case of Morinaga's wave geometry in which $g_{5\lambda} = \delta_{5\lambda}$.

From the equations (4.1), (4.3) and (4.4) we obtain the following differential equation for ψ.

$$\nabla_\mu\psi = R_\mu\psi\,, \tag{4.5}$$

where the operator ∇_μ is defined by

$$\nabla_\mu \equiv \frac{\partial}{\partial x^\mu} + \Lambda_\mu\,.$$

From the fact that the parallelism of X^ν must be invariant by all the transformations (3.7), we know that ∇_μ is an invariant operator; namely by the C-transformation

$$\bar{\nabla}_\mu = \frac{\partial x^\lambda}{\partial \bar{x}^\mu}\nabla_\lambda\,, \qquad \bar{\Lambda}_\mu = \frac{\partial x^\lambda}{\partial \bar{x}^\mu}\Lambda_\mu\,,$$

and by the S-transformation

$$\bar{\nabla}_\mu = \frac{\partial}{\partial \bar{x}^\mu} + \bar{\Lambda}_\mu = S\nabla_\mu S^{-1}\,,$$

$$\bar{\Lambda}_\mu = S\Lambda_\mu S^{-1} + S\frac{\partial S^{-1}}{\partial x^\mu}\,,$$

and R_ν is covariant for C-transformation and is invariant for S-transformation. Hence *the equation (4.5) which we are going to take as the fundamental equation for ψ in our geometry, is invariant for all the transformations (3.7)*. This equation is determined by the microscopic projective metric γ_a, the coefficients of projective connection $\Gamma_{\alpha\beta}^\gamma$ and the multiplier of the parallel displacement Λ_μ.

§ 5. The condition for integrability of the fundamental equation for ψ.

As shown before, when γ_a, $\Gamma_{\alpha\beta}^\gamma$ and Λ_μ are given, the fundamental equation for ψ becomes

$$\frac{\partial \Psi}{\partial x^\mu} = (-\Lambda_\mu + R_\mu I)\Psi .$$

If we calculate the condition for integrability of the above equation, we have the following equations analogously to Morinaga's (6.3) and (6.4),

$$\left.\begin{aligned} &\left[\frac{1}{4}R_{\mu\nu\alpha\beta}\gamma^{[\alpha}\gamma^{\beta]}+f_{\mu\nu}I\right]\Psi = 0 , \\ &\left[\frac{1}{4}R_{\mu\nu\alpha\beta,\,\rho\sigma\ldots}\gamma^{[\alpha}\gamma^{\beta]}+f_{\mu\nu,\,\rho\sigma\ldots}I\right]\Psi = 0 , \end{aligned}\right\} \tag{5.1}$$

where
$$f_{\mu\nu} = \frac{\partial}{\partial x^\nu}(R_\mu - L_\mu) - \frac{\partial}{\partial x^\mu}(R_\nu - L_\nu) ,$$

and $R_{\mu\nu\alpha\beta}$ is the curvature tensor derived from $\Gamma^\gamma_{\alpha\beta}$, and the notation (,) indicates covariant differentiation with respect to the coefficients of connection $\Gamma^\gamma_{\alpha\beta}-\delta^\gamma_\alpha R_\beta$.

Substituting (2.3) into (5.1), we have

$$\left.\begin{aligned} &\left[\frac{1}{4}R_{\lambda\mu}^{\cdot\cdot\alpha\beta}h^\rho_{[\alpha}h^\sigma_{\beta]}\mathring{\gamma}_\rho\mathring{\gamma}_\sigma+f_{\lambda\mu}I\right]\bar{\Psi} = 0 , \\ &\left[\frac{1}{4}R_{\lambda\mu}^{\cdot\cdot\alpha\beta}{}_{,\,\nu\ldots}h^\rho_{[\alpha}h^\sigma_{\beta]}\mathring{\gamma}_\rho\mathring{\gamma}_\sigma+f_{\lambda\mu,\,\nu\ldots}I\right]\bar{\Psi} = 0 , \end{aligned}\right\} \tag{5.2}$$

where
$$\bar{\Psi} = U^{-1}\Psi .$$

If we substitute the actual value of $\mathring{\gamma}_\rho\mathring{\gamma}_\sigma$ and rewrite the above equations, the first becomes

$$\begin{pmatrix} i\overset{3}{k}_{\lambda\mu}-i\overset{5}{k}_{\lambda\mu}+2f_{\lambda\mu}, & i\overset{1}{k}_{\lambda\mu}-\overset{2}{k}_{\lambda\mu}-\overset{6}{k}_{\lambda\mu}-i\overset{8}{k}_{\lambda\mu}, & -\overset{4}{k}_{\lambda\mu}+i\overset{10}{k}_{\lambda\mu}, & i\overset{7}{k}_{\lambda\mu}-\overset{9}{k}_{\lambda\mu} \\ i\overset{1}{k}_{\lambda\mu}+\overset{2}{k}_{\lambda\mu}+\overset{6}{k}_{\lambda\mu}-i\overset{8}{k}_{\lambda\mu}, & -i\overset{3}{k}_{\lambda\mu}+i\overset{5}{k}_{\lambda\mu}+2f_{\lambda\mu}, & i\overset{7}{k}_{\lambda\mu}+\overset{9}{k}_{\lambda\mu}, & -\overset{4}{k}_{\lambda\mu}-i\overset{10}{k}_{\lambda\mu} \\ \overset{4}{k}_{\lambda\mu}+i\overset{10}{k}_{\lambda\mu}, & i\overset{7}{k}_{\lambda\mu}-\overset{9}{k}_{\lambda\mu} & -i\overset{3}{k}_{\lambda\mu}-i\overset{5}{k}_{\lambda\mu}+2f_{\lambda\mu}, & -i\overset{1}{k}_{\lambda\mu}+\overset{2}{k}_{\lambda\mu}-\overset{6}{k}_{\lambda\mu}-i\overset{8}{k}_{\lambda\mu} \\ i\overset{7}{k}_{\lambda\mu}+\overset{9}{k}_{\lambda\mu}, & \overset{4}{k}_{\lambda\mu}-i\overset{10}{k}_{\lambda\mu}, & -i\overset{1}{k}_{\lambda\mu}-\overset{2}{k}_{\lambda\mu}+\overset{6}{k}_{\lambda\mu}-i\overset{8}{k}_{\lambda\mu}, & i\overset{3}{k}_{\lambda\mu}+i\overset{5}{k}_{\lambda\mu}+2f_{\lambda\mu} \end{pmatrix}\begin{pmatrix}\bar{\Psi}_1\\ \bar{\Psi}_2\\ \bar{\Psi}_3\\ \bar{\Psi}_4\end{pmatrix} = 0 , \tag{5.3}$$

where

$$\left.\begin{aligned}
&\overset{1}{k}_{\lambda\mu} = R_{\lambda\mu}^{\cdot\cdot\alpha\beta} h^{1}_{[\alpha} h^{2}_{\beta]}, \quad \overset{2}{k}_{\lambda\mu} = R_{\lambda\mu}^{\cdot\cdot\alpha\beta} h^{1}_{[\alpha} h^{3}_{\beta]}, \quad \overset{3}{k}_{\lambda\mu} = R_{\lambda\mu}^{\cdot\cdot\alpha\beta} h^{1}_{[\alpha} h^{4}_{\beta]}, \\
&\overset{4}{k}_{\lambda\mu} = R_{\lambda\mu}^{\cdot\cdot\alpha\beta} h^{1}_{[\alpha} h^{0}_{\beta]}, \quad \overset{5}{k}_{\lambda\mu} = R_{\lambda\mu}^{\cdot\cdot\alpha\beta} h^{2}_{[\alpha} h^{3}_{\beta]}, \quad \overset{6}{k}_{\lambda\mu} = R_{\lambda\mu}^{\cdot\cdot\alpha\beta} h^{2}_{[\alpha} h^{4}_{\beta]}, \\
&\overset{7}{k}_{\lambda\mu} = R_{\lambda\mu}^{\cdot\cdot\alpha\beta} h^{2}_{[\alpha} h^{0}_{\beta]}, \quad \overset{8}{k}_{\lambda\mu} = R_{\lambda\mu}^{\cdot\cdot\alpha\beta} h^{3}_{[\alpha} h^{4}_{\beta]}, \quad \overset{9}{k}_{\lambda\mu} = R_{\lambda\mu}^{\cdot\cdot\alpha\beta} h^{3}_{[\alpha} h^{0}_{\beta]}, \\
&\overset{10}{k}_{\lambda\mu} = R_{\lambda\mu}^{\cdot\cdot\alpha\beta} h^{4}_{[\alpha} h^{0}_{\beta]},
\end{aligned}\right\}$$

and the second takes the form analogous to (5.3) in which $k_{\lambda\mu}$ is replaced by $k_{\lambda\mu,\nu\ldots}$. For brevity's sake, we shall express (5.3) and the similar equation mentioned above in the following notations.

$$\Delta_{\lambda\mu}\Psi = 0\,,$$

$$\Delta_{\lambda\mu,\nu\ldots}\Psi = 0\,.$$

In order that (5.3) should admit non-vanishing solutions Ψ, it is necessary that

$$\text{det.}\,|\,\Delta_{\lambda\mu}\,| = 0\,, \qquad \text{det.}\,|\,\Delta_{\lambda\mu,\nu\ldots}\,| = 0\,, \qquad \text{etc.}$$

But it has not yet been possible to obtain these conditions in more concrete form, so that we shall leave it for future research and here confine ourselves to the following two simpler cases.(1)

Case 1. When

$$\overset{3}{k}_{\lambda\mu} = \overset{5}{k}_{\lambda\mu}, \quad \overset{1}{k}_{\lambda\mu} = \overset{8}{k}_{\lambda\mu}, \quad \overset{2}{k}_{\lambda\mu} = -\overset{6}{k}_{\lambda\mu}, \quad \overset{4}{k}_{\lambda\mu} = \overset{7}{k}_{\lambda\mu} = \overset{9}{k}_{\lambda\mu} = \overset{10}{k}_{\lambda\mu} = f_{\lambda\mu} = 0\,. \tag{5.4 a}$$

Case 2. When

$$\overset{3}{k}_{\lambda\mu} = -\overset{5}{k}_{\lambda\mu}, \quad \overset{1}{k}_{\lambda\mu} = -\overset{8}{k}_{\lambda\mu}, \quad \overset{2}{k}_{\lambda\mu} = \overset{6}{k}_{\lambda\mu}, \quad \overset{4}{k}_{\lambda\mu} = \overset{7}{k}_{\lambda\mu} = \overset{9}{k}_{\lambda\mu} = \overset{10}{k}_{\lambda\mu} = f_{\lambda\mu} = 0\,. \tag{5.4 b}$$

Both (5.4 a) and (5.4 b) satisfy the condition

$$\text{det.}\,|\,\Delta_{\lambda\mu}\,| = 0\,,$$

identically. Though we cannot assert that these are the only possible cases, they seem to have important significance as will be seen later.

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(1) These equations are necessarily obtained from the condition that the fundamental equation for ψ is completely integrable for $(\overline{\psi}_1, \overline{\psi}_2)$ or $(\overline{\psi}_3, \overline{\psi}_4)$ respectively.

We can prove by the same method as Morinaga's that *the fundamental equation is integrable when either* (*5.4 a*) *or* (*5.4 b*) *holds*. And these two conditions are not compatible except when the curvature tensor vanishes.

If we rewrite (5.4 a) in the form

$$\left.\begin{aligned} &R_{\lambda\mu}^{\cdot\,\cdot\,\alpha\beta}h^1_{[\alpha}h^4_{\beta]} = R_{\lambda\mu}^{\cdot\,\cdot\,\alpha\beta}h^2_{[\alpha}h^3_{\beta]}\,,\\ &R_{\lambda\mu}^{\cdot\,\cdot\,\alpha\beta}h^1_{[\alpha}h^3_{\beta]} = -R_{\lambda\mu}^{\cdot\,\cdot\,\alpha\beta}h^2_{[\alpha}h^4_{\beta]}\,,\\ &R_{\lambda\mu}^{\cdot\,\cdot\,\alpha\beta}h^1_{[\alpha}h^2_{\beta]} = R_{\lambda\mu}^{\cdot\,\cdot\,\alpha\beta}h^3_{[\alpha}h^4_{\beta]}\,,\\ &R_{\lambda\mu}^{\cdot\,\cdot\,\alpha\beta}h^1_{[\alpha}h^0_{\beta]} = R_{\lambda\mu}^{\cdot\,\cdot\,\alpha\beta}h^2_{[\alpha}h^0_{\beta]} = R_{\lambda\mu}^{\cdot\,\cdot\,\alpha\beta}h^3_{[\alpha}h^0_{\beta]} = R_{\lambda\mu}^{\cdot\,\cdot\,\alpha\beta}h^4_{[\alpha}h^0_{\beta]} = 0\,,\\ &\qquad f_{\lambda\mu} = 0\,, \end{aligned}\right\} \tag{5.5}$$

the first four equations can be put together in one equation

$$R_{\lambda\mu}^{\cdot\,\cdot\,\alpha\beta}h^\rho_{[\alpha}h^\sigma_{\beta]} = \frac{1}{2}\varepsilon_{\rho\sigma\tau\omega\varkappa}R_{\lambda\mu}^{\cdot\,\cdot\,\alpha\beta}h^\tau_{[\alpha}h^\omega_{\beta]}\delta^\varkappa_0\,,$$

or

$$R_{\lambda\mu}^{\cdot\,\cdot\,[\alpha\beta]}h^\rho_\alpha h^\sigma_\beta = \frac{1}{2}\varepsilon_{\rho\sigma\tau\omega\varkappa}R_{\lambda\mu}^{\cdot\,\cdot\,[\alpha\beta]}h^\tau_\alpha h^\omega_\beta\delta^\varkappa_0\,, \tag{5.6}$$

where suffices τ and ω are summed, and the expression $\varepsilon_{\rho\sigma\tau\omega\varkappa}$, when $\rho, \sigma, \tau, \omega, \varkappa$ is a permutation of $0, 1, 2, 3, 4$, has the value 1 or -1 according as the number of the inversion is even or odd and otherwise has the value 0.

Multiplying (5.6) by $h^\rho_\xi h^\sigma_\eta$ and summing for ρ and σ, we have

$$R_{\lambda\mu}^{\cdot\,\cdot\,[\alpha\beta]}\gamma_{\alpha\xi}\gamma_{\beta\eta} = \frac{1}{2}\varepsilon_{\rho\sigma\tau\omega\varkappa}R_{\lambda\mu}^{\cdot\,\cdot\,[\alpha\beta]}h^\rho_\xi h^\sigma_\eta h^\tau_\alpha h^\omega_\beta\delta^\varkappa_0\,. \tag{5.7}$$

Substituting (2.4), i. e.

$$h^\varkappa_0 = \delta^\varkappa_0\,,$$

into (5.7), we have

$$R_{\lambda\mu[\xi\eta]} = \frac{1}{2}D\varepsilon_{\xi\eta\alpha\beta 0}R_{\lambda\mu}^{\cdot\,\cdot\,\alpha\beta}\,,$$

where

$$D = \begin{vmatrix} h^0_0 & \cdots & h^0_4 \\ \vdots & & \vdots \\ h^4_0 & \cdots & h^4_4 \end{vmatrix} = \begin{vmatrix} 1 & \varphi_i \\ 0 & \\ \vdots & p^i_j \\ 0 & \end{vmatrix} = |\,p^i_j\,|\,.$$

Therefore

$$D^2 = |\gamma_{\alpha\beta}| = \begin{vmatrix} 1 & \varphi_i \\ \varphi_i & g_{ij}+\varphi_i\varphi_j \end{vmatrix} = |g_{ij}| = g\,.$$

And finally we obtain the following important equation:

$$R_{\lambda\mu[\xi\eta]} = \frac{\sqrt{g}}{2}\varepsilon_{\xi\eta\alpha\beta0}R_{\lambda\mu}^{\cdot\cdot\,\alpha\beta}\,.\text{(1)} \tag{5.8 a}$$

Conversely, we can easily deduce (5.5) from (5.8 a). Therefore (5.8 a) and $f_{\lambda\mu} = 0$ together make a sufficient condition for integrability of the fundamental equation for ψ. Similarly from (5.4 b), we have

$$R_{\lambda\mu[\xi\eta]} = -\frac{\sqrt{g}}{2}\varepsilon_{\xi\eta\alpha\beta0}R_{\lambda\mu}^{\cdot\cdot\,\alpha\beta}\,. \tag{5.8 b}$$

Also (5.8 a) and (5.8 b) are written in the following form, respectively.[2]

$$R_{\lambda\mu}^{\cdot\cdot\,[\xi\eta]} = \pm\frac{1}{2\sqrt{g}}\varepsilon^{\xi\eta\alpha\beta\varkappa}R_{\lambda\mu\alpha\beta}\varphi_\varkappa\,, \tag{F}$$

$$\varepsilon^{\xi\eta\alpha\beta\varkappa} = \varepsilon_{\xi\eta\alpha\beta\varkappa}\,.$$

Further we make the assumption that *the connection of our projective space satisfies*

$$G_{\alpha\beta|\rho} = 0\,.$$

Then the coefficients of the connection $\Gamma^\gamma_{\alpha\beta}$ become the Christoffel symbols derived from $G_{\alpha\beta}$, i. e.

$$\Gamma^\gamma_{\alpha\beta} = \left\{{}^{\,\gamma}_{\alpha\beta}\right\}_{G_{\alpha\beta}}\,. \tag{5.9}$$

Then from (2.1), we know that this connection is obtained from the Riemannian by a conformal transformation. Therefore, the curvature tensor satisfies the relation

$$\left.\begin{aligned} &R_{(\omega\mu)\lambda\nu} = 0\,, \qquad R_{\omega\mu(\lambda\nu)} = 0\,, \qquad R_{[\omega\mu\lambda]\nu} = 0\,, \\ &R_{\omega\mu\lambda\nu} = R_{\lambda\nu\omega\mu}\,, \\ &R_{\mu\lambda} = R_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\omega} = R_{\lambda\mu}\,, \qquad \text{etc.} \end{aligned}\right\} \tag{5.10}$$

(1) $\sqrt{g}$ means one of the two values of $g^{\frac{1}{2}}$ which coincides with $|p^i_j|$.

(2) See **Note 1** at the end of this paper, p. 164.

In this case, (F) becomes

$$R_{\lambda\mu}^{\cdot\cdot\xi\eta} = \frac{\pm 1}{2\sqrt{g}} \varepsilon^{\xi\eta\alpha\beta\varkappa} R_{\lambda\mu\alpha\beta}\varphi_{\varkappa} .^{(1)} \tag{5.11}$$

Contracting the above for λ and η, we have

$$R_{\mu}^{\cdot\xi} = \pm\frac{1}{2\sqrt{g}} \varepsilon^{\xi\eta\alpha\beta\varkappa} R_{\alpha\beta\eta\mu}\varphi_{\varkappa} = \pm\frac{1}{2\sqrt{g}} \varepsilon^{\xi\eta\alpha\beta\varkappa} R_{[\alpha\beta\eta]\mu}\varphi_{\varkappa} = 0 .$$

Hence we have

$$R_{\lambda\mu} = 0 . \tag{F$_C$}$$

§ 6. The contracted fundamental equation (F$_C$).

Since we regard the projective tensor equation (F) as the fundamental equation, it is urgently necessary to express this equation in ordinary tensor-form. For brevity's sake, we will consider the case of plus sign only. The relations among the coefficients of connections, namely $\Gamma^{\gamma}_{\alpha\beta} \equiv \{^{\gamma}_{\alpha\beta}\}_{G_{\alpha\beta}}$, $\{^{\gamma}_{\alpha\beta}\} \equiv \{^{\gamma}_{\alpha\beta}\}_{\gamma_{\alpha\beta}}$ and $\{^{k}_{ij}\}_{g_{ij}}$, have been given by Veblen[2] as follows:

$$\Gamma^{\alpha}_{\beta\gamma} = \{^{\alpha}_{\beta\gamma}\} + (\delta^{\alpha}_{\beta}\Phi_{\gamma} + \delta^{\alpha}_{\gamma}\Phi_{\beta} - \gamma_{\beta\gamma}\Phi^{\alpha}) \tag{6.1}$$

$$\left.\begin{aligned}
&\{^{\alpha}_{00}\} = 0 , \\
&\{^{\alpha}_{\beta 0}\} = \gamma^{i\alpha}\varphi_{i\beta} , \qquad \text{that is} \qquad \{^{i}_{j0}\} = \varphi^{i}_{\cdot j} , \qquad \{^{0}_{j0}\} = -\varphi_i\varphi^{i}_{\cdot j} , \\
&\{^{i}_{jk}\} = \{^{i}_{jk}\}_{g_{ij}} + \varphi^{i}_{\cdot j}\varphi_k + \varphi^{i}_{\cdot k}\varphi_j , \\
&\{^{0}_{jk}\} = -\varphi_i\{^{i}_{jk}\} + \frac{1}{2}\left(\frac{\partial\varphi_j}{\partial x^k} + \frac{\partial\varphi_k}{\partial x^j}\right), \qquad \varphi_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial\varphi_\alpha}{\partial x^\beta} - \frac{\partial\varphi_\beta}{\partial x^\alpha}\right),
\end{aligned}\right\} \tag{6.2}$$

---

(1) In the above discussion, we began with (5.2) in order to obtain the fundamental equation. When we begin with the equivalent equation

$$\left.\begin{aligned}
&\left[\frac{1}{4}R_{\lambda\mu[\alpha\beta]}\bar{h}^{\alpha}_{\rho}\bar{h}^{\beta}_{\sigma}\mathring{\gamma}^{\rho}\mathring{\gamma}^{\sigma} + f_{\lambda\mu}I\right]\bar{\Psi} = 0 , \\
&\left[\frac{1}{4}R_{\lambda\mu[\alpha\beta],\nu\ldots}\bar{h}^{\alpha}_{\rho}\bar{h}^{\beta}_{\sigma}\mathring{\gamma}^{\rho}\mathring{\gamma}^{\sigma} + f_{\lambda\mu,\nu\ldots}I\right]\bar{\Psi} = 0 ,
\end{aligned}\right\}$$

in place of (5.2), using the relation

$$\delta^{\varkappa}_{0} = h^{\varkappa}_{0} = \gamma_{0\alpha}\bar{h}^{\alpha}_{\varkappa} = \varphi_{\alpha}\bar{h}^{\alpha}_{\varkappa} ,$$

we can arrive at the same result (5.11). The + and − signs of (5.11) correspond to the + and − signs of (F) respectively.

(2) O. Veblen, *Projektive Relativitätstheorie*, 44, 49.

where Φ_a is a projective vector of index 0 defined by the equation

$$\Phi_a = \frac{\partial \log \Phi}{\partial x^a}.$$

To rewrite projective equations in affine form we use the following principle. If $M_{\alpha\beta\gamma\ldots}$ is a projective covariant tensor, the transformation (3.1) being a special form, the quantities

$$\left.\begin{aligned} M^{ijk\cdots} &= \gamma^{i\alpha}\gamma^{j\beta}\gamma^{k\gamma}\cdots M_{\alpha\beta\gamma\ldots}, \\ M^{ij\cdot k\cdots}_{\cdot\cdot 0\cdot} &= \gamma^{i\alpha}\gamma^{j\beta}\gamma^{k\delta}\cdots M_{\alpha\beta 0\delta\ldots}, \\ M^{i\cdot\cdot j\cdots}_{\cdot 00\cdot} &= \gamma^{i\alpha}\gamma^{j\beta}\cdots M_{\alpha 00\beta\ldots}, \\ M_{000\ldots}, &\qquad \text{etc.,} \end{aligned}\right\}$$

are all affine tensors of the nature indicated by their suffices.

First, let us express the contracted fundamental equation by an affine tensor equation. We denote the curvature tensor made from $\{^{\alpha}_{\beta\gamma}\}$ by $B_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$, then

$$R_{\lambda\mu\xi}^{\cdot\cdot\cdot\sigma} = B_{\lambda\mu\xi}^{\cdot\cdot\cdot\sigma} + \delta^{\sigma}_{\lambda}(\Phi_{\xi;\mu} - \Phi_{\xi}\Phi_{\mu} + \Phi^{\epsilon}\Phi_{\epsilon}\gamma_{\xi\mu}) - \delta^{\sigma}_{\mu}(\Phi_{\xi;\lambda} - \Phi_{\xi}\Phi_{\lambda} + \Phi^{\epsilon}\Phi_{\epsilon}\gamma_{\xi\lambda})$$
$$-\gamma_{\xi\lambda}(\Phi^{\sigma}_{;\mu} - \Phi^{\sigma}\Phi_{\mu}) + \gamma_{\xi\mu}(\Phi^{\sigma}_{;\lambda} - \Phi^{\sigma}\Phi_{\lambda}), \tag{6.3}$$

$$B_{\alpha\beta} = K_{ij}\delta^i_{\alpha}\delta^j_{\beta} - \varphi^{s}_{\cdot i,s}(\delta^i_{\alpha}\varphi_{\beta} + \delta^i_{\beta}\varphi_{\alpha}) + 2\varphi^{s}_{\cdot\beta}\varphi_{s\alpha} + \varphi^{s}_{\cdot t}\varphi^{t}_{\cdot s}\varphi_{\alpha}\varphi_{\beta}, \tag{6.4}$$

where $K_{ijk}^{\cdot\cdot\cdot l}$ is the Riemannian curvature tensor made from $\{^{i}_{jk}\}_{g_{ij}}$, the notations $(;)$ and $(,)$ indicate covariant differentiations with respect to $\{^{\alpha}_{\beta\gamma}\}$ and $\{^{i}_{jk}\}_{g_{ij}}$ respectively, and the upper and lower Greek suffices are transmitted by $\gamma_{\alpha\beta}$ and $\gamma^{\alpha\beta}$, and the upper and lower Latin ones by g_{ij} and g^{ij}.

From (F_C), (6.3) and (6.4), after some calculation, *we have the following three equations as the contracted fundamental equations.*[1]

$$K^{ij} - \frac{1}{2}g^{ij}(K - 2\lambda) = -2(E^{ij} + M^{ij}) \tag{I'}$$

$$J^i = 3[\varphi^{i}_{\cdot j}\theta^j - N\theta^i] \tag{II'}$$

$$3e - 2m - 4H = 0 \tag{III'}$$

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(1) See **Note 2**, p. 165.

where

$$E^{ij} = \frac{1}{4} g^{ij}\varphi^{t}_{\cdot s}\varphi^{s}_{\cdot t} + g^{st}\varphi^{i}_{\cdot s}\varphi^{j}_{\cdot t} = e^{ij} - \frac{1}{4} g^{ij} e\,, \tag{6.5}$$

$$M^{ij} = -\frac{3}{2}\left[g^{h(j}\theta^{i)}_{,h} + \theta^{i}\theta^{j} - g^{ij}(\theta^{h}_{,h} - \theta^{h}\theta_{h})\right] = m^{ij} - g^{ij} m\,, \tag{6.6}$$

$$J^{i} = \varphi^{is}_{,s}\,, \tag{6.7}$$

$$e^{ij} = g^{st}\varphi^{i}_{\cdot s}\varphi^{j}_{\cdot t}\,, \qquad e = g_{ij}e^{ij}\,, \tag{6.8}$$

$$m^{ij} = -\frac{3}{2}\left[g^{h(j}\theta^{i)}_{,h} + \theta^{i}\theta^{j}\right] + g^{ij}H\,, \qquad m = g_{ij}m^{ij}\,, \qquad H = \theta^{i}\theta_{i}\,, \tag{6.9}$$

$$\theta_{a} = N\varphi_{a} - \Phi_{a}\,, \tag{6.10}$$

$$N = \sqrt{-\frac{\lambda}{3}}\,. \tag{6.11}$$

From (6.5), (6.6), (6.8) and (6.9) we see that all of the four tensors E^{ij}, M^{ij}, e^{ij} and m^{ij} are symmetric. The above quantities E^{ij}, J^{i} and M^{ij} can pure formally be interpreted as the electromagnetic energy tensor, charge and current vector and the energy tensor, respectively.

Specially, when $\theta_i = 0$, from (6.10) we have $\varphi_{ij} = 0$, therefore

$$e^{ij} = m^{ij} = 0\,,$$

so

$$E^{ij} = M^{ij} = 0\,,$$

and the equation (I′) becomes

$$K^{ij} = \lambda g^{ij}\,,$$

where λ is the cosmological constant. Thus *we have pure formally the Einstein and de Sitter law of gravitation with cosmological constant* λ. In this case the other two equations become identities.

When N is very small and Φ is a function of x^0 only, from (6.10), we have $\theta_i \doteqdot 0$, but φ_{ij} does not necessarily vanish. Then we have the fundamental equation of the form

$$\left.\begin{aligned} &K^{ij}-\frac{1}{2}g^{ij}K+2E^{ij}=0\,, \\ &J^{i}=0\,, \\ &K=0\,, \\ \text{and}\qquad &e=0\,. \end{aligned}\right\}$$

The first three of the above formally coincide with Veblen's result.[1] But in our case we also have the fourth equation.

§ 7. The fundamental equation (F).

Next, in this section we shall investigate the affine form of the fundamental equation (F). In the previous section, we saw that we can construct sixteen affine tensors from the curvature tensor $R_{\lambda\mu\xi\eta}$, but from the relation (5.10) we know that only three of these tensors are essentially independent, viz. R^{ijkl}, $R^{ijk}{}_{\cdot\cdot\cdot 0}$ and $R^{i\cdot j}_{\cdot 0\cdot 0}$. In the equation

$$R_{\lambda\mu\xi\eta}=\frac{\sqrt{g}}{2}\varepsilon_{\xi\eta\alpha\beta0}R_{\lambda\mu}^{\cdot\cdot\alpha\beta}\,, \tag{5.8 a}$$

if we multiply by $\gamma^{i\lambda}\gamma^{j\mu}\gamma^{k\xi}\gamma^{l\eta}$ and sum for λ, μ, ξ, η, we have the following affine tensor equation.[2]

$$\frac{1}{2\sqrt{g}}\varepsilon^{klpq}G_{pq}^{\cdot\cdot ij}=G^{klij}\,, \tag{I}$$

where $G_{pq}^{\cdot\cdot ij}$ is an affine tensor defined by

$$G^{klij}\equiv K^{klij}+2E^{klij}+2M^{klij}+2NH^{klij}+\frac{1}{3}\Big(\lambda-\frac{K}{2}\Big)(g^{kj}g^{li}-g^{lj}g^{ki})\,,$$

where

$$2E^{klij}=2\varphi^{kl}\varphi^{ji}-\varphi^{jk}\varphi^{li}+\varphi^{jl}\varphi^{ki}+\frac{1}{2}(-g^{kj}e^{il}+g^{lj}e^{ik}+g^{ki}e^{jl}-g^{li}e^{jk})\,, \tag{7.1 a}$$

$$2M^{klij}=m(-g^{kj}g^{il}+g^{lj}g^{ik})-(-g^{kj}m^{il}+g^{lj}m^{ik}+g^{ki}m^{jl}-g^{li}m^{jk})\,, \tag{7.1 b}$$

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(1) O. Veblen, *Projektive Relativitätstheorie*, 52.

(2) See **Note 3, p. 166.**

$$2H^{klij} = -g^{kj}\varphi^{il}+g^{lj}\varphi^{ik}+g^{ki}\varphi^{jl}-g^{li}\varphi^{jk}, \tag{7.1 c}$$

in which e^{ij}, m^{ij}, e and m are quantities defined in (6.8) and (6.9). As can be readily seen, (I) can also be written in the following form.

$$\frac{\sqrt{g}}{2}\varepsilon_{stpq}G^{pq}_{\cdot\cdot lm} = G_{stlm}.$$

From these equations we can easily deduce the following relations:

$$E^{ka\cdot j}_{\cdot\cdot a} = E^{kj}, \tag{7.2 a}$$

$$M^{ka\cdot j}_{\cdot\cdot a} = M^{kj}, \tag{7.2 b}$$

$$H^{ka\cdot j}_{\cdot\cdot a} = \varphi^{kj}, \tag{7.2 c}$$

which correspond exactly to the relation,

$$K^{ka\cdot j}_{\cdot\cdot a} = K^{kj}, \tag{7.2 d}$$

in the Riemannian curvature tensor. Here E^{kj} and M^{kj} are symmetric tensors defined in (6.5) and (6.6). Furthermore, from (7.1 a), (7.1 b) and (7.1 c) we have the following equations analogously to (5.10):

$$E_{(kl)ij} = 0, \quad E_{kl(ij)} = 0, \quad E_{klij} = E_{ijkl} \quad \text{and} \quad E_{[kli]j} = 0, \tag{7.3 a}$$

$$M_{(kl)ij} = 0, \quad M_{kl(ij)} = 0, \quad M_{klij} = M_{ijkl} \quad \text{and} \quad M_{[kli]j} = 0, \tag{7.3 b}$$

$$H_{(kl)ij} = 0, \quad H_{kl(ij)} = 0, \quad H_{klij} = -H_{ijkl} \quad \text{and} \quad H_{[kli]j} = g_{j[k}\varphi_{li]}. \tag{7.3 c}$$

The equations (7.3 a) and (7.3 b) show that both E_{ijkl} and M_{ijkl} satisfy a complete set of identities of the curvature tensor of a metric space.[1]

Contracting (I) for i and l and taking its symmetric part, we obtain (I′), the contracted fundamental equation,[2] and from the antisymmetric part we have[3]

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(1) T. Thomas, *The Differential Invariants of Generalized Space*, Cambridge (1934), 131.

(2), (3) See **Note 4**, p. 169.

$$\frac{1}{2\sqrt{g}}\varepsilon^{kjlp}\varphi_{lp} = \varphi^{kj} \qquad \text{or} \qquad \frac{\sqrt{g}}{2}\varepsilon_{stpq}\varphi^{pq} = \varphi_{st}. \tag{7.4}$$

This equation may be considered as giving the wave property as will be seen at the end of this section.

Multiplying (5.8 a) by $\gamma^{i\lambda}\gamma^{j\mu}\gamma^{k\xi}$, summing for λ, μ and ξ, and putting $\eta = 0$, we have(1)

$$J^{kji} - J^{kij} = g^{jk}\varphi^{i}_{\cdot h}\theta^h - g^{ik}\varphi^{j}_{\cdot h}\theta^h + N(g^{ik}\theta^j - g^{jk}\theta^i), \tag{II}$$

where $$J_{kji} = -J_{jki} = \varphi_{kj,i}\,.^{(2)}$$

If we contract (II) for i and k, we obtain (II′).

Multiplying (5.8 a) by $\gamma^{i\mu}\gamma^{j\xi}$, summing for μ and ξ and putting $\lambda = \eta = 0$ we obtain the third relation connecting e^{ij} and m^{ij}.(3)

$$3e^{ij} - 2m^{ij} - g^{ij}H = 0. \tag{III}$$

The relation (III′) is obtained from this by contracting for i and j.

When $\theta^i = \varphi^{ij} = e^{ij} = m^{ij} = E^{ij} = M^{ij} = 0$, the equation (I) becomes as follows:

$$\frac{1}{2\sqrt{g}}\left[\varepsilon^{klpq}K_{pq}^{\cdot\cdot ij} + \frac{2}{3}\lambda\varepsilon^{klij}\right] = K^{klij} - \frac{\lambda}{3}(g^{kj}g^{il} - g^{lj}g^{ik}). \tag{7.5}$$

And in this case, the other two equations (II) and (III) become identities. Hence in this case the fundamental equation is (7.5) alone, and if we contract it for i and l, we have the equation

$$K^{ij} = \lambda g^{ij}.$$

Morinaga's equation (1.3) *can be obtained by neglecting the constant* λ *in our equation* (7.5).

We have introduced (I), (II) and (III) from the equation (5.8 a), and conversely, if these three equations are satisfied, we can easily

(1) See **Note 5**, p. 169.

(2) As J_i is obtained by contracting $\varphi_{kj,i}$ for i and j, we here adopt the notation J_{kji} instead of $\varphi_{kj,i}$, which may, as it were, be considered as corresponding to the charge and current tensor.

(3) See **Note 6**, p. 170.

deduce (5.8 a). Hence, (I), (II), (III) and $f_{\lambda\mu} = 0$[1] together is a sufficient condition for integrability of the fundamental equation for ψ.

If we neglect λ in (I) from the beginning, we can get rid of the equation (7.4), but in general this equation should not be ommitted. Its physical meaning has not yet been fully given, but the following may at least be said: That, if we adopt Cartesian coordinates x^i in the case of Euclidean space, this equation can be written in the form:

$$D\varphi = 0\,,^{(2)} \tag{7.6}$$

where φ is a 1–4 matrix whose elements are φ_i, and D is a differential operator satisfying the relation

$$D^{+}D = \Box\,;$$

so that from (7.6) we have

$$\Box\,\varphi = 0\,,$$

which is the equation of propagation of φ in empty space. The equation (7.6) just corresponds to Maxwell's electromagnetic equation in vacuum,

that is[3] $$D\mathfrak{F} = 0\,.$$

§ 8. Notes.

Note 1.

From (5.8 a) we have

$$R_{\lambda\mu[\xi\eta]} = \frac{1}{2}\frac{g}{\sqrt{g}}\varepsilon_{\xi\eta\alpha\beta 0}R_{\lambda\mu}^{\cdot\cdot\,\alpha\beta}$$

hence $$R_{\lambda\mu[\xi\eta]} = \frac{1}{2\sqrt{g}}\varepsilon^{\rho\sigma\tau\omega\varkappa}\gamma_{\xi\rho}\gamma_{\eta\sigma}\gamma_{\alpha\tau}\gamma_{\beta\omega}\gamma_{0\varkappa}R_{\lambda\mu}^{\cdot\cdot\,\alpha\beta}$$

therefore we have

$$R_{\lambda\mu}^{\cdot\cdot\,[\xi\eta]} = \frac{1}{2\sqrt{g}}\varepsilon^{\xi\eta\alpha\beta\varkappa}R_{\lambda\mu\alpha\beta}\varphi_{\varkappa}\,.$$

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(1) Since L_μ is also an arbitrary vector, this condition does not violate the arbitrarity of R_μ.

(2) See **Note 7**, p. 171 and T. Sibata, This Journal, **5** (1935), 189.

(3) Y. Mimura and T. Iwatsuki, loc. cit., 212.

Note 2.

From (6.3) and (6.4) we have

$$\left.\begin{aligned} R_{\mu\xi} &= B_{\mu\xi}+(n-1)(\Phi_{\mu;\xi}-\Phi_\mu\Phi_\xi+\Phi^\sigma\Phi_\sigma\gamma_{\mu\xi})+\gamma_{\mu\xi}\Phi^\sigma_{;\sigma}\,, \\ B^{lm} &= \gamma^{l\alpha}\gamma^{m\beta}B_{\alpha\beta} = K^{lm}+2g^{sh}\varphi^l_{\cdot s}\varphi^m_{\cdot h}\,, \qquad (n=4) \end{aligned}\right\}\ \text{(N2.1)}$$

respectively. From these two equations and (Fc) we have

$$K^{ij}+2g^{st}\varphi^i_{\cdot s}\varphi^j_{\cdot t}+(n-1)(g^{ij}F-\theta^i\theta^j+\Phi^i_{;\xi}\gamma^{\xi j})+g^{ij}\Phi^\sigma_{;\sigma} = 0\,.$$

On the other hand from (6.2) and (6.11) we have

$$\gamma^{\xi j}\Phi^i_{;\xi} = N\varphi^{ij}-\theta^i_{,h}g^{jh}\,, \qquad N = \sqrt{-\frac{\lambda}{3}}$$

hence $K^{ij}+2g^{st}\varphi^i_{\cdot s}\varphi^j_{\cdot t}+3(g^{ij}F-\theta^{ij}+N\varphi^{ij})+g^{ij}A = 0\,,$

where $F = \Phi^\sigma\Phi_\sigma = N^2+\theta^i\theta_i\,, \qquad \theta_{ij} = \theta_{i,j}+\theta_i\theta_j\,,$

$$A = \Phi^\sigma_{;\sigma} = -\theta^i_{,i} = -\frac{1}{\sqrt{g}}\frac{\partial(\theta^i\sqrt{g})}{\partial x^i}\,. \qquad \text{(N2.2)}$$

The antisymmetric part of (N2.2) is reduced to the identity $N\varphi_{ij} = \theta_{[i,j]}$ and the symmetric part becomes

$$K^{ij}+2g^{st}\varphi^i_{\cdot s}\varphi^j_{\cdot t}+3(g^{ij}F-\theta^{(ij)})+g^{ij}A = 0\,. \qquad \text{(N2.3)}$$

But from (N2.1) and (6.4) we have

$$B^i_{\cdot 0}+3[N\theta^i-\varphi^i_j\theta^j] = 0\,, \qquad B^i_{\cdot 0} = \varphi^{is}_{,s}\,.$$

From this we can get one of the equations sought, i. e. (II′). In like manner from (N2.1) and (6.4) we have

$$B_{00}+3\theta_h\theta^h-\theta^h_{,h} = \varphi^s_{\cdot t}\varphi^t_{\cdot s}+3\theta_h\theta^h-\theta^h_{,h} = 0\,.$$

This is nothing but (III′).

Now for the sake of brevity we put

$$F = N^2+H\,, \qquad H = \theta^i\theta_i\,. \qquad \varphi^s_{\cdot t}\varphi^t_{\cdot s} = -e\,. \qquad \text{(N2.4)}$$

If we contract (N2.3) for i and j and use (N2.4), we have

$$K = -5e-4A+4\lambda\,.$$

From these equations, (N2.3) becomes

$$K^{ij}-\frac{1}{2}g^{ij}(K-2\lambda)+2g^{st}\varphi^{i}_{\cdot s}\varphi^{j}_{\cdot t}-g^{ij}\Big(\frac{1}{2}e+3H+3A\Big)-3\theta^{(ij)}=0\,, \tag{N2.5}$$

from which (I′) can be readily obtained.

Note 3.

Multiplying (5.8 a) by $\gamma^{i\lambda}\gamma^{j\mu}\gamma^{k\xi}\gamma^{l\eta}$ and summing for λ, μ, ξ and η we obtain

$$\gamma^{i\lambda}\gamma^{j\mu}\gamma^{k\xi}\gamma^{l\eta}R_{\lambda\mu\xi\eta}=\frac{\sqrt{g}}{2}\epsilon_{pqrs}\gamma^{i\lambda}\gamma^{j\mu}\gamma^{kp}\gamma^{lq}\gamma^{r\rho}\gamma^{s\sigma}R_{\lambda\mu\rho\sigma}\,. \tag{N3.1}$$

Now let us calculate the left hand side of the above. From (6.3) we have

$$\gamma^{i\lambda}\gamma^{j\mu}\gamma^{k\xi}\gamma^{l\eta}R_{\lambda\mu\xi\eta}=\gamma^{i\lambda}\gamma^{j\mu}\gamma^{k\xi}\gamma^{l\eta}B_{\lambda\mu\xi\eta}+(\delta^{j}_{\xi}A^{i}_{\eta}-\delta^{j}_{\eta}A^{i}_{\xi}-\delta^{i}_{\xi}B^{j}_{\eta}+\delta^{i}_{\eta}B^{j}_{\xi})\gamma^{k\xi}\gamma^{l\eta}\,, \tag{N3.2}$$

where $\quad A^{i}_{\xi}=\Phi^{i}_{;\xi}-\Phi^{i}\Phi_{\xi}+F\delta^{i}_{\xi}\quad$ and $\quad B^{i}_{\xi}=\Phi^{i}_{;\xi}-\Phi^{i}\Phi_{\xi}\,.$

From (6.2) we have

$$\Phi^{i}_{;0}=-\theta^{h}\varphi^{i}_{\cdot h}\,,\qquad \Phi^{i}_{;a}=-\theta^{i}_{,a}-\theta^{h}\varphi^{i}_{\cdot h}\varphi_{a}\,,$$

hence

$$\left.\begin{aligned}A^{i}_{0}&=-\varphi^{i}_{\cdot h}\theta^{h}+N\theta^{i}=B^{i}_{0}\,,\\ A^{i}_{a}&=-\theta^{i}_{,a}+\theta^{i}(N\varphi_{a}-\theta_{a})+F\delta^{i}_{a}-\theta^{h}\varphi^{i}_{\cdot h}\varphi_{a}=B^{i}_{a}+F\delta^{i}_{a}\,.\end{aligned}\right\} \tag{N3.3}$$

Now we will calculate $\gamma^{i\lambda}\gamma^{j\mu}\gamma^{k\xi}\gamma^{l\eta}B_{\lambda\mu\xi\eta}$ in (N3.2).

$$\begin{aligned}\gamma^{i\lambda}\gamma^{j\mu}\gamma^{k\xi}\gamma^{l\eta}B_{\lambda\mu\xi\eta}&=\gamma^{i\lambda}\gamma^{k\xi}\gamma^{l\eta}B_{\xi\eta\lambda}^{\cdot\cdot\cdot j}\\ &=\gamma^{i\lambda}\gamma^{k\xi}\gamma^{l\eta}\Big(\frac{\partial}{\partial x^{\eta}}\left\{{j \atop \lambda\xi}\right\}-\frac{\partial}{\partial x^{\xi}}\left\{{j \atop \lambda\eta}\right\}-\left\{{j \atop \tau\xi}\right\}\left\{{\tau \atop \lambda\eta}\right\}+\left\{{j \atop \tau\eta}\right\}\left\{{\tau \atop \lambda\xi}\right\}\Big)\,.\end{aligned}$$

Using (6.2) and the relation $\gamma^{ij}=g^{ij}$, this becomes

$$\begin{aligned}&=g^{kb}g^{la}g^{ic}\frac{\partial}{\partial x^{a}}\left\{{j \atop cb}\right\}+\gamma^{k0}g^{la}g^{ic}\frac{\partial}{\partial x^{a}}\left\{{j \atop c0}\right\}+g^{kb}g^{la}\gamma^{i0}\frac{\partial}{\partial x^{a}}\left\{{j \atop 0b}\right\}\cdots\cdots A_{1}\\ &\quad-g^{kb}g^{la}g^{ic}\frac{\partial}{\partial x^{b}}\left\{{j \atop ca}\right\}-\gamma^{l0}g^{kb}g^{ic}\frac{\partial}{\partial x^{b}}\left\{{j \atop c0}\right\}-g^{kb}g^{la}\gamma^{i0}\frac{\partial}{\partial x^{b}}\left\{{j \atop 0a}\right\}\cdots\cdots A_{2}\\ &\quad-\gamma^{k\xi}\gamma^{l\eta}\gamma^{i\lambda}\Big(\underset{\dot{B}_1}{\left\{{j \atop 0\xi}\right\}\left\{{0 \atop \lambda\eta}\right\}}+\underset{\dot{B}_2}{\left\{{j \atop m\xi}\right\}\left\{{m \atop \lambda\eta}\right\}}-\underset{\dot{B}_3}{\left\{{j \atop 0\eta}\right\}\left\{{0 \atop \lambda\xi}\right\}}-\underset{\dot{B}_4}{\left\{{j \atop m\eta}\right\}\left\{{m \atop \lambda\xi}\right\}}\Big)\,.\end{aligned}$$

Substituting (6.2) into the above we have

$$A_1 = g^{kb}g^{la}g^{ic}\frac{\partial}{\partial x^a}\{{}^{\,j}_{cb}\}_g + g^{kb}g^{la}\varphi^{ji}\frac{\partial\varphi_b}{\partial x^a} + g^{la}g^{ic}\varphi^{jk}\frac{\partial\varphi_c}{\partial x^a}$$

$$-A_2 = g^{kb}g^{la}g^{ic}\frac{\partial}{\partial x^b}\{{}^{\,j}_{ca}\}_g + g^{kb}g^{la}\varphi^{ji}\frac{\partial\varphi_a}{\partial x^b} + g^{kb}g^{ic}\varphi^{jl}\frac{\partial\varphi_c}{\partial x^b}$$

so

$$A_1 + A_2 = g^{kb}g^{la}g^{ic}\left[\frac{\partial}{\partial x^a}\{{}^{\,j}_{cb}\}_g - \frac{\partial}{\partial x^b}\{{}^{\,i}_{ca}\}_g\right] + 2\varphi^{ji}\varphi^{kl} + g^{ic}\frac{\partial\varphi_c}{\partial x^a}[g^{la}\varphi^{jk} - g^{ka}\varphi^{jl}].$$

In like manner we have

$$B_1 = -\varphi^{jk}\{\varphi^{i,l} + \varphi^{li}\}, \qquad B_3 = \varphi^{jl}\{\varphi^{i,k} + \varphi^{ki}\},$$

and
$$\left.\begin{aligned} -B_2 &= g^{kb}g^{la}g^{ic}\{{}^{\,j}_{mb}\}_g\{{}^{m}_{ca}\}_g + \varphi^{jk}\varphi_m g^{la}g^{ic}\{{}^{m}_{ca}\}_g, \\ B_4 &= g^{kb}g^{la}g^{ic}\{{}^{\,j}_{ma}\}_g\{{}^{m}_{cb}\}_g + \varphi^{jl}\varphi_m g^{kb}g^{ic}\{{}^{m}_{cb}\}_g, \end{aligned}\right\}$$

where, for brevity's sake, though φ_i is not an affine vector, we have put

$$\varphi^{i,l} = g^{ia}g^{lb}\varphi_{a,b} = g^{ia}g^{lb}\left(\frac{\partial\varphi_a}{\partial x^b} - \{{}^{\,c}_{ab}\}_g\varphi_c\right).$$

From these equations we finally obtain

$$\gamma^{i\lambda}\gamma^{j\mu}\gamma^{k\xi}\gamma^{l\eta}B_{\xi\mu\lambda\eta} = g^{kb}g^{la}g^{ic}K_{bac}{}^{\cdot\cdot\cdot j} + 2\varphi^{ji}\varphi^{kl} - \varphi^{jk}\{\varphi^{i,l} + \varphi^{li}\} + \varphi^{jl}\{\varphi^{i,k} + \varphi^{ki}\}$$
$$+ g^{ic}\frac{\partial\varphi_c}{\partial x^a}\{g^{la}\varphi^{jk} - g^{ka}\varphi^{jl}\} - \varphi^{jk}\varphi_m g^{la}g^{ic}\{{}^{m}_{ca}\}_g + \varphi^{jl}\varphi_m g^{kb}g^{ic}\{{}^{m}_{cb}\}_g$$
$$= K^{klij} + 2\varphi^{kl}\varphi^{ji} - \varphi^{jk}\varphi^{li} + \varphi^{jl}\varphi^{ki}. \qquad \text{(N3.4)}$$

Next we will calculate the remaining terms of the right hand side of (N3.2).

$$\gamma^{k\xi}\gamma^{l\eta}(\delta^j_\xi A^i_\eta - \delta^j_\eta A^i_\xi) = g^{kj}(\gamma^{l0}A^i_0 + g^{la}A^i_a) - g^{lj}(\gamma^{k0}A^i_0 + g^{kb}A^i_b).$$

Substituting A's into the above from (N3.3), and using the relation[1]

$$\theta^{il} = -\frac{2}{3}m^{il} + \frac{2}{3}g^{il}H + N\varphi^{il}, \qquad \text{(N3.5)}$$

we have

(1) This relation is obtained from (6.9), (6.10) and (N2.2).

$$\gamma^{k\xi}\gamma^{l\eta}(\delta^j_\xi A^i_\eta-\delta^j_\eta A^i_\xi) = F(g^{kj}g^{li}-g^{lj}g^{ki})-\frac{2}{3}(-g^{kj}m^{il}+g^{lj}m^{ik})$$
$$+\frac{2}{3}H(-g^{kj}g^{il}+g^{lj}g^{ik})+N(-g^{kj}\varphi^{il}+g^{lj}\varphi^{ik})$$
$$=\frac{1}{3}(H-\lambda)(g^{kj}g^{li}-g^{lj}g^{ki})-\frac{2}{3}(-g^{kj}m^{il}+g^{lj}m^{ik})$$
$$+N(-g^{kj}\varphi^{il}+g^{lj}\varphi^{ik}). \tag{N3.6}$$

In like manner we have

$$-\gamma^{k\xi}\gamma^{l\eta}(\delta^i_\xi B^j_\eta-\delta^i_\eta B^j_\xi) = -\frac{2}{3}(g^{ki}m^{jl}-g^{li}m^{jk})+\frac{2}{3}H(g^{ki}g^{jl}-g^{li}g^{jk})$$
$$-N(-g^{ki}\varphi^{jl}+g^{li}\varphi^{jk}). \tag{N3.7}$$

From (N3.4), (N3.6) and (N3.7) we have

$$\gamma^{i\lambda}\gamma^{j\mu}\gamma^{k\xi}\gamma^{l\eta}R_{\lambda\mu\xi\eta}$$
$$=K^{klij}+2E^{klij}+2NH^{klij}+\frac{1}{3}\left(\lambda-\frac{K}{2}\right)(g^{kj}g^{li}-g^{lj}g^{ki})$$
$$-\frac{2}{3}(-g^{kj}m^{il}+g^{lj}m^{ik}+g^{ki}m^{jl}-g^{li}m^{jk})$$
$$+\left(-\frac{2}{3}\lambda+\frac{K}{6}-\frac{H}{3}\right)(g^{kj}g^{li}-g^{lj}g^{ki})$$
$$-\frac{1}{2}(-g^{kj}e^{il}+g^{lj}e^{ik}+g^{ki}e^{jl}-g^{li}e^{jk}). \tag{N3.8}$$

But from (III) we have(1)

$$e^{ij}=\frac{2}{3}m^{ij}+\frac{H}{3}g^{ij}.$$

Substituting this relation into the last term of (N3.8) we have

$$\gamma^{i\lambda}\gamma^{j\mu}\gamma^{k\xi}\gamma^{l\eta}R_{\lambda\mu\xi\eta}=K^{klij}+2E^{klij}+2NH^{klij}+\frac{1}{3}\left(\lambda-\frac{K}{2}\right)(g^{kj}g^{li}-g^{lj}g^{ki})$$
$$-(-g^{kj}m^{il}+g^{lj}m^{ik}+g^{ki}m^{jl}-g^{li}m^{jk})+\frac{1}{6}(-4\lambda+K)(g^{kj}g^{li}-g^{lj}g^{ki}). \tag{N3.9}$$

---

(1) See **Note 6**, p. 170 and note that we can deduce this relation independently of (I).

But if we rewrite (III′) we have $3H-e+A=0$, hence from (N2.4) we have

$$-4\lambda+K=-5e-4A=-9e+12H=-6m. \qquad \text{(N3.10)}$$

From (N3.9) and (N3.10) we finally obtain

$$\gamma^{i\lambda}\gamma^{j\mu}\gamma^{k\xi}\gamma^{l\eta}R_{\lambda\mu\xi\eta}=G^{klij}. \qquad \text{(N3.11)}$$

In the next place, we will calculate the right hand side of (N3.1). Using (N3.11) we have

$$\frac{\sqrt{g}}{2}\varepsilon_{pqrs0}\gamma^{i\lambda}\gamma^{j\mu}\gamma^{kp}\gamma^{lq}\gamma^{r\rho}\gamma^{s\sigma}R_{\lambda\mu\rho\sigma}=\frac{\sqrt{g}}{2}\varepsilon_{pqrs0}g^{kp}g^{lq}G^{rsij}$$

$$=\frac{\sqrt{g}}{2}\varepsilon_{pqrs0}g^{kp}g^{lq}g^{ra}g^{sb}G_{ab}^{\cdot\cdot ij}=\frac{1}{2\sqrt{g}}\varepsilon^{klpq}G_{pq}^{\cdot\cdot ij}. \qquad \text{(N3.12)}$$

From (N3.11) and (N3.12) we can obtain (I).

Note 4.

From (7.2 a), (7.2 b), (7.2 c) and (7.2 d) we have

$$g_{il}G^{klij}=K^{kj}+2E^{kj}+2M^{kj}+2N\varphi^{kj}+\left(\lambda-\frac{K}{2}\right)g^{kj},$$

and from (7.3 a), (7.3 b) and (7.3 c) we have

$$g_{il}\varepsilon^{klpq}G_{pq}^{\cdot\cdot ij}=\varepsilon^{klpq}G_{[pql]}^{\cdot\cdot\cdot j}=2N\varepsilon^{kjpq}\varphi_{pq}.$$

Contracting (I) for i and l and using the above relations we have

$$\frac{N}{\sqrt{g}}\varepsilon^{kjpq}\varphi_{pq}=K^{kj}+2E^{kj}+2M^{kj}+2N\varphi^{kj}+\left(\lambda-\frac{K}{2}\right)g^{kj}. \qquad \text{(N4.1)}$$

Taking the symmetric and antisymmetric parts of (N4.1) we can easily obtain (I′) and (7.4) respectively.

Note 5.

In the equation (5.8 a), if we put $\eta=0$, the right hand side vanishes so we have

$$R_{\lambda\mu\xi0}=0, \qquad \text{hence} \qquad R^{ijk}_{\cdot\cdot\cdot 0}=0.$$

On the other hand from (6.3) we have

$$R_{\lambda\mu\xi 0} = B_{\lambda\mu\xi 0}+\varphi_\lambda(\Phi_{\xi;\mu}-\Phi_\xi\Phi_\mu+F\gamma_{\xi\mu})-\varphi_\mu(\Phi_{\xi;\lambda}-\Phi_\xi\Phi_\lambda+F\gamma_{\xi\lambda})$$
$$-\gamma_{\xi\lambda}(\Phi_{0;\mu}-\Phi_0\Phi_\mu)+\gamma_{\xi\mu}(\Phi_{0;\lambda}-\Phi_0\Phi_\lambda)\,.$$

Multiplying the above by $\gamma^{i\lambda}\gamma^{j\mu}\gamma^{k\xi}$ and summing for λ, μ and ξ we have

$$R^{ijk}_{\cdot\cdot\cdot 0} = B^{ijk}_{\cdot\cdot\cdot 0}-g^{ik}(\Phi_{0;a}g^{ja}-N\Phi^j)+g^{jk}(\Phi_{0;a}g^{ia}-N\Phi^i)\,.$$

Substituting the value of $\Phi_{0;a}$ into this, we have

$$R^{ijk}_{\cdot\cdot\cdot 0} = B^{ijk}_{\cdot\cdot\cdot 0}+g^{ik}\theta^h\varphi^j_{\cdot h}-g^{jk}\theta^h\varphi^i_{\cdot h}-Ng^{ik}\theta^j+Ng^{jk}\theta^i\,.$$

But, as $\left\{{\gamma \atop \alpha\beta}\right\}$ does not include x^0, we have

$$B^{ijk}_{\cdot\cdot\cdot 0} = -\gamma^{i\lambda}\gamma^{j\mu}B_{\lambda\mu 0}^{\cdot\cdot\cdot k} = -\gamma^{i\lambda}\gamma^{j\mu}\left(\frac{\partial}{\partial x^\mu}\left\{{k \atop 0\lambda}\right\}-\frac{\partial}{\partial x^\lambda}\left\{{k \atop 0\mu}\right\}-\left\{{k \atop \rho\lambda}\right\}\left\{{\rho \atop 0\mu}\right\}+\left\{{k \atop \rho\mu}\right\}\left\{{\rho \atop 0\lambda}\right\}\right)$$

$$= -g^{ia}g^{jb}\frac{\partial}{\partial x^b}\left\{{k \atop 0a}\right\}+g^{ia}g^{jb}\frac{\partial}{\partial x^a}\left\{{k \atop 0b}\right\}+\gamma^{i\lambda}g^{jb}\left\{{k \atop \rho\lambda}\right\}\left\{{\rho \atop 0b}\right\}-g^{ia}\gamma^{j\mu}\left\{{k \atop \rho\mu}\right\}\left\{{\rho \atop 0a}\right\}$$

$$= -g^{jb}\varphi^{ki}_{\ \ ,b}+g^{ia}\varphi^{kj}_{\ \ ,a}\,.$$

Then, (II) is readily obtained from these equations.

Note 6.

In the same way as in Note 5, from (5.8 a) we have

$$R_{0\cdot\cdot 0}^{\cdot ij} = 0 \qquad \text{(N6.1)}$$

From (6.3) we have

$$R_{0\mu\xi 0} = B_{0\mu\xi 0}+(\Phi_{\xi;\mu}-\Phi_\xi\Phi_\mu+F\gamma_{\mu\xi})-\varphi_\mu(\Phi_{\xi;0}-\Phi_\xi\Phi_0+F\varphi_\xi)$$
$$-\varphi_\xi(\Phi_{0;\mu}-\Phi_0\Phi_\mu)+\gamma_{\xi\mu}(\Phi_{0;0}-\Phi_0^2)\,.$$

Multiplying this equation by $\gamma^{i\mu}\gamma^{j\xi}$ and summing for μ and ξ we have

$$R_{0\cdot\cdot 0}^{\cdot ij} = B_{0\cdot\cdot 0}^{\cdot ij}+\Phi^j_{;\mu}\gamma^{i\mu}-\theta^i\theta^j+Fg^{ij}-g^{ij}N^2\,. \qquad \text{(N6.2)}$$

After some calculation from (N2.4) and (6.2), we have

$$\gamma^{i\mu}\Phi^j_{;\mu} = -g^{ia}\theta^j_{,a}+N\varphi^{ji}\,, \qquad \text{(N6.3)}$$

hence $$R_{0\cdot\cdot 0}^{\cdot ij} = B_{0\cdot\cdot 0}^{\cdot ij}-g^{ia}\theta^j_{,a}+N\varphi^{ji}-\theta^i\theta^j+g^{ij}H\,,$$

Using the relation $\varphi_\epsilon \left\{ {\epsilon \atop \rho 0} \right\} = \varphi_\epsilon \gamma^{\epsilon i} \varphi_{i\rho} = 0$, we have

$$B^{\cdot ij}_{0\cdot\cdot 0} = \gamma^{i\mu}\gamma^{j\xi}\gamma_{0\epsilon}B^{\cdot\cdot\cdot\epsilon}_{0\mu\xi} = \gamma^{i\mu}\gamma^{j\xi}\varphi_\epsilon\left(\frac{\partial}{\partial x^\mu}\left\{ {\epsilon \atop \xi 0} \right\} - \frac{\partial}{\partial x^0}\left\{ {\epsilon \atop \xi \mu} \right\} - \left\{ {\epsilon \atop \rho 0} \right\}\left\{ {\rho \atop \xi \mu} \right\} + \left\{ {\epsilon \atop \rho \mu} \right\}\left\{ {\rho \atop \xi 0} \right\}\right)$$

$$= g^{ib}g^{ja}\varphi_\epsilon \frac{\partial}{\partial x^b}\left\{ {\epsilon \atop a 0} \right\} + \gamma^{i\mu}g^{ja}\varphi_\epsilon\left\{ {\epsilon \atop \rho \mu} \right\}\left\{ {\rho \atop a 0} \right\} = \varphi^{cj}\varphi^{i}_{\cdot c} = -e^{ij}. \qquad \text{(N6.4)}$$

From (N6.1), (N6.2), (N6.3) and (N6.4) we have

$$-e^{ij} - g^{ia}\theta^j_{,a} + N\varphi^{ji} - \theta^i\theta^j + g^{ij}H = 0.$$

From (N2.2) this becomes

$$e^{ij} + \theta^{(ij)} - g^{ij}H = 0.$$

Hence from (N3.5) we obtain (III).

Note 7.

Rewriting (7.4), we have

$$\varphi_{12} = \sqrt{g}\,\varphi^{34}, \qquad \varphi_{13} = -\sqrt{g}\,\varphi^{24}, \qquad \varphi_{14} = \sqrt{g}\,\varphi^{23}.$$

In the case of Cartesian coordinates this becomes

$$\varphi_{12} = \varphi_{34}, \qquad \varphi_{13} = -\varphi_{24}, \qquad \varphi_{14} = \varphi_{23}. \qquad \text{(N7.1)}$$

Since φ_i is determined except for gradient vector and this vector is quite arbitrary, we can put

$$\text{div.}\,\varphi = 0, \qquad \text{(N7.2)}$$

in order to remove this arbitrarity. Rewriting (N7.1) and (N7.2) in actual form we have

$$\left.\begin{aligned}
\frac{\partial\varphi_1}{\partial x^1} + \frac{\partial\varphi_2}{\partial x^2} + \frac{\partial\varphi_3}{\partial x^3} + \frac{\partial\varphi_4}{\partial x^4} &= 0 \\
-\frac{\partial\varphi_1}{\partial x^2} + \frac{\partial\varphi_2}{\partial x^1} + \frac{\partial\varphi_3}{\partial x^4} - \frac{\partial\varphi_4}{\partial x^3} &= 0 \\
\frac{\partial\varphi_1}{\partial x^3} + \frac{\partial\varphi_2}{\partial x^4} - \frac{\partial\varphi_3}{\partial x^1} - \frac{\partial\varphi_4}{\partial x^2} &= 0 \\
\frac{\partial\varphi_1}{\partial x^4} - \frac{\partial\varphi_2}{\partial x^3} + \frac{\partial\varphi_3}{\partial x^2} - \frac{\partial\varphi_4}{\partial x^1} &= 0
\end{aligned}\right\}$$

that is $$D\varphi = 0\,, \qquad D = \sigma^i \frac{\partial}{\partial x^i}$$

where σ's are constant matrices used by G. Rumer.[1]

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(1) T. Sibata, loc. cit., 191.