On Systems of Simultaneous Functional Equations.

By

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The object of this paper is to search for a system of functions that are linearly transformed when their variables are linearly transformed.

(I)

Problem: Under the assumptions that

(i) The equation (1.1) below is satisfied independently of x's and a's,

(ii) $w^{\varkappa\lambda}(x)$, $K^{\varkappa\lambda}_{\mu\nu}(a)$ have first partial derivatives,

to solve the following functional equation

$$w^{\varkappa\lambda}(X) = \sum_{\mu,\nu} K^{\varkappa\lambda}_{\mu\nu}(a)\cdot w^{\mu\nu}(x)\,, \qquad (\varkappa, \lambda = 1, 2, \ldots, n) \tag{1.1}$$

where

$$X^i = \sum_r a^i_r x^r\,, \qquad (i = 1, 2, \ldots, n)$$

and $w^{\varkappa\lambda}(x)$, $K^{\varkappa\lambda}_{\mu\nu}(a)$ represent respectively functions of x's and a's.

Solution: Differentiating (1.1) with a^i_i, we have

$$\frac{\partial w^{\varkappa\lambda}(X)}{\partial X^i}\cdot\frac{\partial X^i}{\partial a^i_i} = \sum_{\mu,\nu}\frac{\partial K^{\varkappa\lambda}_{\mu\nu}(a)}{\partial a^i_i}\cdot w^{\mu\nu}(x)\,. \tag{1.2}$$

Put $a^i_j = \delta^i_j$ = Kronecker's delta, then we have a system of differential equations with respect to x^i,

$$x^i\frac{\partial w^{\varkappa\lambda}(x)}{\partial x^i} = \sum_{\mu,\nu} L^{\varkappa\lambda}_{\mu\nu}\cdot w^{\mu\nu}(x)\,, \tag{1.3}$$

where

$$L^{\varkappa\lambda}_{\mu\nu} = \left[\frac{\partial K^{\varkappa\lambda}_{\mu\nu}}{\partial a^i_i}\right]_{a=\delta}.$$

For the time being, i is considered as being fixed. (1.3) reduces, by the change of variable $x^i = e^{\xi^i}$, to

$$\frac{\partial W^{\kappa\lambda}(\xi^i)}{\partial \xi^i} = \sum L^{\kappa\lambda}_{\mu\nu} \cdot W^{\mu\nu}(\xi^i)\,, \tag{1.4}$$

where $$W^{\kappa\lambda}(\xi^i) = w^{\kappa\lambda}(e^{\xi^i})\,.$$

The system of simultaneous linear differential equations of constant coefficients (1.4) has as its solution

$$W^{\kappa\lambda}(\xi^i) = g_1^{\kappa\lambda}(\xi^i)e^{r_1\xi^i} + g_2^{\kappa\lambda}(\xi^i)e^{r_2\xi^i} + \ldots\ldots, \tag{1.5}$$

where $r_1, r_2, \ldots\ldots$ are μ_1-ple, μ_2-ple, ... roots respectively of the characteristic equation of the given system,

$$f(r) = \begin{vmatrix} L^{11}_{11} - r & L^{11}_{12} & \ldots & L^{11}_{nn} \\ L^{12}_{11} & L^{12}_{12} - r & \ldots & L^{12}_{nn} \\ \vdots & & & \vdots \\ L^{nn}_{11} & L^{nn}_{12} & \ldots & L^{nn}_{nn} - r \end{vmatrix} = 0\,,$$

and $g_i^{\kappa\lambda}$ is any polynomial of $(\mu_i - 1)$th degree at highest.[1] Thus we have

$$w^{\kappa\lambda}(x^i) = (x^i)^{r_1} g_1^{\kappa\lambda}(\log x^i) + (x^i)^{r_2} g_2^{\kappa\lambda}(\log x^i) + \ldots\ldots$$

Now put

$$(a^i_j) = A = \begin{pmatrix} 1 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 1 & \ddots & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ a^i_1 & a^i_2 & \ldots & a^i_i & \ldots & a^i_n \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & 1 \end{pmatrix},$$

then we have

$$X^i = a^i_i x^i + b \qquad \text{where} \qquad b = \sum_{r \neq i} a^i_r x^r\,,$$

and $$X^r = x^r \qquad (r \neq i)\,.$$

Thus with regard to x^i, (1.1) becomes

(1) Horn, *Gewöhnliche Differentialgleichungen*, 84.

(1.6)

$$(a_i^i x^i + b)^{r_1} \cdot g_1^{\varkappa\lambda}\big(\log (a_i^i x^i + b)\big) + (a_i^i x^i + b)^{r_2} \cdot g_2^{\varkappa\lambda}\big(\log (a_i^i x^i + b)\big) + \dots\dots$$
$$= \sum_{\mu,\nu} K_{\mu\nu}^{\varkappa\lambda}\{(x^i)^{r_1} \cdot g_1^{\mu\lambda}(\log x^i) + (x^i)^{r_2} \cdot g_2^{\mu\nu}(\log x^i) + \dots\dots\}\,.$$

Since the terms consist of elementary functions, (1.6) must hold not only for real but also for complex values of x^i. If some $\mu_i > 1$ or some r_i is neither zero nor a positive integer, the singular point of the right-hand side is $x^i = 0$ while that of the left-hand side is $-\dfrac{b}{a_i^i}$ which may be thought to be non-zero without any loss of generality. Thus g's can not contain x^i, and r's are all zero or positive integers. Hence

With regard to any variable x^i, $w^{\varkappa\lambda}(x)$ must to be a polynomial; and the number of its terms can not exceed n^2.

Thus finally we see easily that $w^{\varkappa\lambda}(x)$ *must be a polynomial of all its variables.*

Now we can put

$$w^{\varkappa\lambda}(x) = A^{\varkappa\lambda} + \sum_i A_i^{\varkappa\lambda} x^i + \sum_{i,j} A_{ij}^{\varkappa\lambda} x^i \cdot x^j + \sum_{i,j,k} A_{ijk}^{\varkappa\lambda} \cdot x^i \cdot x^j \cdot x^k + \dots\dots$$

where A's are assumed to be symmetric with respect to their lower suffices. Then (1.1) becomes

$$A^{\varkappa\lambda} + \sum_i A_i^{\varkappa\lambda}(\sum_r a_r^i x^r) + \sum_{i,j} A_{ij}^{\varkappa\lambda}(\sum_r a_r^i x^r)(\sum_s a_s^j x^s) + \dots\dots$$
$$\equiv \sum_{\mu,\nu} K_{\mu\nu}^{\varkappa\lambda}\{A^{\mu\nu} + \sum_i A_i^{\mu\nu} \cdot x^i + \sum_{i,j} A_{ij}^{\varkappa\lambda} \cdot x^i \cdot x^j + \dots\dots\}\,.$$

$$\therefore \begin{cases} \sum_{\mu,\nu} K_{\mu\nu}^{\varkappa\lambda} \cdot A^{\mu\nu} = A^{\varkappa\lambda} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (0) \\ \sum_{\mu,\nu} K_{\mu\nu}^{\varkappa\lambda} \cdot A_i^{\mu\nu} = \sum_r A_r^{\varkappa\lambda} \cdot a_i^r \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \qquad \vdots \\ \sum K_{\mu\nu}^{\varkappa\lambda} \cdot A_{ij\dots k}^{\mu\nu} = \sum A_{rs\dots t}^{\varkappa\lambda} \cdot a_i^r \cdot a_j^s \dots a_k^t \dots\dots\dots (p) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \qquad \vdots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (m) \end{cases}$$

$K_{\mu\nu}^{\varkappa\lambda}$ must be determined so that the conditions (0), (1), ... (p), ... are satisfied if the suitable choice of A's allows it.

In order that $K_{\mu\nu}^{\varkappa\lambda}$'s may be determined from (p), A's must be such that the ranks of

(1.7)

$$\begin{pmatrix} A_{11\ldots1}^{11} & \ldots & A_{11\ldots1}^{nn} \\ \vdots & & \vdots \\ A_{nn\ldots n}^{11} & \ldots & A_{nn\ldots n}^{nn} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_{11\ldots1}^{11} & \ldots & A_{11\ldots1}^{nn} & \sum A_{rs\ldots t}^{\varkappa\lambda} \cdot a_1^r \cdot a_1^s \ldots a_1^t \\ \vdots & & \vdots & \vdots \\ A_{nn\ldots n}^{11} & \ldots & A_{nn\ldots n}^{nn} & \sum A_{rst\ldots t}^{\varkappa\lambda} \cdot a_n^r \cdot a_n^s \ldots a_n^t \end{pmatrix}$$

are coincident (with r say), independently of the values of a's. This is possible when and only when,

(i) $r = {}_nH_p = \dfrac{n(n+1)\ldots.(n+p-1)}{1\cdot 2\ldots.p}$ = the number of equations in (p),

or

(ii) $\quad r < {}_nH_p \quad$ and $\quad A_{rs\ldots t}^{\varkappa\lambda} = 0\,.$

Proof: (i) This is quite clear.

(ii) There exists a determinant of order $r+1$

$$(1.8) \qquad \begin{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \vdots & \vdots \\ \ldots\ldots\ldots\ldots\ldots & \sum A_{rs\ldots t}^{\varkappa\lambda} a_i^r a_j^s \ldots a_k^t \end{vmatrix} = 0$$

where the cofactor of $\sum A_{rs\ldots t}^{\varkappa\lambda} a_i^r a_j^s \ldots. a_k^t$ is non-vanishing. Expanding (1.8) with respect to the last column, we have

$$(1.9) \quad (\sum A_{rs\ldots t}^{\varkappa\lambda} \cdot a_i^r \cdot a_j^s \ldots. a_k^t) \times (\text{non-vanishing determinant}) + \ldots. = 0\,.$$

Since this expression is an identity with respect to a's and moreover, since there cannot be such a term as $a_i^r a_j^s \ldots. a_k^t$ with definite lower suffices $i, j, \ldots., k$ in (1.9) other than the first, we have

$$\sum A_{rs\ldots t}^{\varkappa\lambda} a_i^r \cdot a_j^s \ldots. a_k^t \times (\text{non-vanishing determinant}) = 0\,.$$

Now since A's are symmetric with respect to their lower suffices, we have for any $r, s, \ldots., t$ $A_{rs\ldots t}^{\varkappa\lambda} = 0$ as required.

Thus if $p \geqq 4$, as we have

$$ {}_nH_p > n^2 \geqq r , $$

all A's vanish which means that the terms of higher degree than the third cannot appear; in other words, p must be equal to or less than three.

Case I, $p = 3$.

In ${}_nH_3 \geqq n^2 \geqq r$, the left equality holds only for $n = 2$.

Terms of third degree can appear when and only when $n = 2$, *and then those which appear in respective* $w^{\varkappa\lambda}(x)$ *must be linearly independent of each other,* since the rank r is equal to ${}_nH_p$.

From (3) we also see that

$K^{\varkappa\lambda}_{\mu\nu} = a$ *H. E.*[1] *of* $a^1_1, a^2_1, a^1_2, a^2_2$ *of third degree.*

Substituting this result in (0), (1), (2) respectively we have a H. E. of third degree in the left-hand sides of the respective equalities, whose right-hand sides, being a constant, a H. E. of first degree and a H. E. of second degree respectively, we must have $A^{\varkappa\lambda} = 0$, $A^{\varkappa\lambda}_r = 0$ and $A^{\varkappa\lambda}_{rs} = 0$. Thus we arrive at the following result:—

Terms of third degree can appear in $w^{\varkappa\lambda}$*'s when and only when* $n = 2$, *and then* $w^{11}, w^{12}, w^{21}, w^{22}$ *are H. E. of third degree which are linearly independent of each otker.* $K^{\varkappa\lambda}$*'s are also H. E. of third degree of* a*'s determined by* (3).

Case II $m \leqq 2$

$H = {}_nH_0 + {}_nH_1 + {}_nH_2 = \dfrac{(n+1)(n+2)}{2}$ = the total number of equations in (0), (1), (2) $\lesseqgtr n^2$, according as $n \geqq 4$ or $n < 4$.

If $n \geqq 4$, $w^{\varkappa\lambda}$ may be composed of terms of second and first degree and an absolute term, since A's may be determined so that the linear simultaneous eqations (0), (1), (2) can be solved. $K^{\varkappa\lambda}_{\mu\nu}$'s are also H. E. of similar form about a's.

If $n = 3$, we must examine the ranks of the following two matrices obtained from (0), (1), (2),

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(1) H. E. = homogeneous expression.

$$\begin{pmatrix} A^{11} & A^{12} & A^{21} & A^{13} & A^{31} & A^{22} & A^{23} & A^{32} & A^{33} \\ A^{11}_{1} & A^{12}_{1} & A^{21}_{1} & A^{13}_{1} & A^{31}_{1} & A^{22}_{1} & A^{23}_{1} & A^{32}_{1} & A^{33}_{1} \\ A^{11}_{2} & A^{12}_{2} & A^{21}_{2} & A^{13}_{2} & A^{31}_{2} & A^{22}_{2} & A^{23}_{2} & A^{32}_{2} & A^{33}_{2} \\ A^{11}_{3} & A^{12}_{3} & A^{21}_{3} & A^{13}_{3} & A^{31}_{3} & A^{22}_{3} & A^{23}_{3} & A^{32}_{3} & A^{33}_{3} \\ A^{11}_{11} & A^{12}_{11} & A^{21}_{11} & A^{13}_{11} & A^{31}_{11} & A^{22}_{11} & A^{23}_{11} & A^{32}_{11} & A^{33}_{11} \\ A^{11}_{12} & A^{12}_{12} & A^{21}_{12} & A^{13}_{12} & A^{31}_{12} & A^{22}_{12} & A^{23}_{12} & A^{32}_{12} & A^{33}_{12} \\ A^{11}_{13} & A^{12}_{13} & A^{21}_{13} & A^{13}_{13} & A^{31}_{13} & A^{22}_{13} & A^{23}_{13} & A^{32}_{13} & A^{33}_{13} \\ A^{11}_{22} & A^{12}_{22} & A^{21}_{22} & A^{13}_{22} & A^{31}_{22} & A^{22}_{22} & A^{23}_{22} & A^{32}_{22} & A^{33}_{22} \\ A^{11}_{23} & A^{12}_{23} & A^{21}_{23} & A^{13}_{23} & A^{31}_{23} & A^{22}_{23} & A^{23}_{23} & A^{32}_{23} & A^{33}_{23} \\ A^{11}_{33} & A^{12}_{33} & A^{21}_{33} & A^{13}_{33} & A^{31}_{33} & A^{22}_{33} & A^{23}_{33} & A^{32}_{33} & A^{33}_{33} \end{pmatrix}$$

and

$$\begin{pmatrix} A^{11} & A^{12} & A^{21} & A^{13} & A^{31} & A^{22} & A^{23} & A^{32} & A^{33} & A^{\varkappa\lambda} \\ A^{11}_{1} & A^{12}_{1} & A^{21}_{1} & A^{13}_{1} & A^{31}_{1} & A^{22}_{1} & A^{23}_{1} & A^{32}_{1} & A^{33}_{1} & \sum A^{\varkappa\lambda}_{r} \cdot a^{r}_{1} \\ A^{11}_{2} & A^{12}_{2} & A^{21}_{2} & A^{13}_{2} & A^{31}_{2} & A^{22}_{2} & A^{23}_{2} & A^{32}_{2} & A^{33}_{2} & \sum A^{\varkappa\lambda}_{r} \cdot a^{r}_{2} \\ A^{11}_{3} & A^{12}_{3} & A^{21}_{3} & A^{13}_{3} & A^{31}_{3} & A^{22}_{3} & A^{23}_{3} & A^{32}_{3} & A^{33}_{3} & \sum A^{\varkappa\lambda}_{r} \cdot a^{r}_{3} \\ A^{11}_{11} & A^{12}_{11} & A^{21}_{11} & A^{13}_{11} & A^{31}_{11} & A^{22}_{11} & A^{23}_{11} & A^{32}_{11} & A^{33}_{11} & \sum A^{\varkappa\lambda}_{rs} \cdot a^{r}_{1} \cdot a^{s}_{1} \\ A^{11}_{12} & A^{12}_{12} & A^{21}_{12} & A^{13}_{12} & A^{31}_{12} & A^{22}_{12} & A^{23}_{12} & A^{32}_{12} & A^{33}_{12} & \sum A^{\varkappa\lambda}_{rs} \cdot a^{r}_{1} \cdot a^{s}_{2} \\ A^{11}_{13} & A^{12}_{13} & A^{21}_{12} & A^{13}_{13} & A^{31}_{13} & A^{22}_{13} & A^{23}_{13} & A^{32}_{13} & A^{33}_{13} & \sum A^{\varkappa\lambda}_{rs} \cdot a^{r}_{1} \cdot a^{s}_{3} \\ A^{11}_{22} & A^{12}_{22} & A^{21}_{22} & A^{13}_{22} & A^{31}_{22} & A^{22}_{22} & A^{23}_{22} & A^{32}_{22} & A^{33}_{22} & \sum A^{\varkappa\lambda}_{rs} \cdot a^{r}_{2} \cdot a^{s}_{2} \\ A^{11}_{23} & A^{12}_{23} & A^{21}_{23} & A^{13}_{23} & A^{31}_{23} & A^{22}_{23} & A^{23}_{23} & A^{32}_{23} & A^{33}_{23} & \sum A^{\varkappa\lambda}_{rs} \cdot a^{r}_{2} \cdot a^{s}_{3} \\ A^{11}_{33} & A^{12}_{33} & A^{21}_{33} & A^{13}_{33} & A^{31}_{33} & A^{22}_{33} & A^{23}_{33} & A^{32}_{33} & A^{33}_{33} & \sum A^{\varkappa\lambda}_{rs} \cdot a^{r}_{3} \cdot a^{s}_{3} \end{pmatrix}$$

Similar consideration as before leads us to

$$A^{\varkappa\lambda} \times (\text{non-vanishing determinant}) + \ldots. = 0$$

or $$\sum A^{\varkappa\lambda}_{r} a^{r}_{i} \times (\text{non-vanishing determinant}) + \ldots. = 0$$

or $$\sum A^{\varkappa\lambda}_{rs} \cdot a^{r}_{i} \cdot a^{s}_{j} \times (\text{non-vanishing determinant}) + \ldots. = 0\,;$$

from which we get $A^{\kappa\lambda}=0$, or $A_r^{\kappa\lambda}=0$ or $A_{rs}^{\kappa\lambda}=0$ respectively. Thus we obtain the result that

If $n=3$, $w^{\kappa\lambda}$'s must be one of the following three forms,
(*A*) *a H.E. of first degree + a H.E. of second degree,*
(*B*) *a H.E. of first dagree + an absolute term,*
(*C*) *a H.E. of second degree + an absolute term.*
In each case, we can assign A's so that (0), (1), (2) *dertermine K's.*
$K_{\mu r}^{\kappa\lambda}$'s are also of the same corresponding forms respectively.

If $n=2$, a similar examination of

$$\begin{pmatrix} A^{11} & A^{12} & A^{21} & A^{22} \\ A_1^{11} & A_1^{12} & A_1^{21} & A_1^{22} \\ A_2^{11} & A_2^{12} & A_2^{21} & A_2^{22} \\ A_{11}^{11} & A_{11}^{12} & A_{11}^{21} & A_{11}^{22} \\ A_{12}^{11} & A_{12}^{12} & A_{12}^{21} & A_{12}^{22} \\ A_{22}^{11} & A_{22}^{12} & A_{22}^{21} & A_{22}^{22} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A^{11} & A^{12} & A^{21} & A^{22} & A^{\kappa\lambda} \\ A_1^{11} & A_1^{12} & A_1^{21} & A_1^{22} & \sum A_r^{\kappa\lambda}\cdot a_1^r \\ A_2^{11} & A_2^{12} & A_2^{21} & A_2^{22} & \sum A_r^{\kappa\lambda}\cdot a_2^r \\ A_{11}^{11} & A_{11}^{12} & A_{11}^{21} & A_{11}^{22} & \sum A_{rs}^{\kappa\lambda}\cdot a_1^r\cdot a_1^s \\ A_{12}^{11} & A_{12}^{12} & A_{12}^{21} & A_{12}^{22} & \sum A_{rs}^{\kappa\lambda}\cdot a_1^r\cdot a_2^s \\ A_{22}^{11} & A_{22}^{12} & A_{22}^{21} & A_{22}^{22} & \sum A_{rs}^{\kappa\lambda}\cdot a_2^r\cdot a_2^s \end{pmatrix}$$

shows that $w^{\kappa\lambda}$ must at least have the same forms as in the case $n=3$, but it is easily seen that the first case (*A*) cannot be sufficient, thus we arrive at the following result

If $n=2$, $w^{\kappa\lambda}$ must be composed either of
(*A*) *a H.E. of first degree + an absolute term, or of*
(*B*) *a H.E. of second degree + an absolute term.*
K's are also of the corresponding forms.

(II)

Problem: To solve the functional equation

$$w^{\kappa\lambda}(X;Y)=\sum_{\mu,\nu}K_{\mu\nu}^{\kappa\lambda}(a)\cdot w^{\mu\nu}(x;y) \tag{2.1}$$

where
$$X^i=\sum_{r=1}^n a_r^i x^r, \qquad Y^i=\sum_{r=1}^n a_r^i y^r$$

under the same assumptions as in (I).

Solution: Put $y^r=\alpha x^r$, then we have $Y^i=\alpha X^i$, and this gives rise to no change in $K^{\varkappa\lambda}_{\mu\nu}(a)$. Again put

$$w^{\varkappa\lambda}(x^1, x^2, \ldots\ldots, x^n\,;\, ax^1, ax^2, \ldots\ldots, ax^n) \equiv V^{\varkappa\lambda}(x^1, x^2, \ldots\ldots, x^n)\,,$$

then we have

$$V^{\varkappa\lambda}(X) = \sum K^{\varkappa\lambda}_{\mu\nu} V^{\mu\nu}(x)\,. \tag{2.2}$$

If $V^{\varkappa\lambda}(x) \not\equiv 0$, that is to say, if it is not the case where $w^{\varkappa\lambda}(x\,;y)$ always vanishes when x and y are colinear with the origin, then the proof given in (I) shows that

If $n = 2$, $K^{\varkappa\lambda}_{\mu\nu}$ must be equal to one of the following three forms,
(A) a H. E. of third degree,
(B) (a H. E. of second degree + a constant),
(C) (a H. E. of first degree + a constant).

If $n = 3$, $K^{\varkappa\lambda}_{\mu\nu}$ must be equal to one of
(A) (a H. E. of second degree + a constant),
(B) (a H. E. of first degree + a constant),
(C) (a H. E. of second degree + a H. E. of first degree).

If $n \geqq 4$, $K^{\varkappa\lambda}_{\mu\nu}$ must be equal to
(a H.E. of second degree + a H.E. of first degree + a constant).

Thus it becomes clear from (2.1) that $w^{\varkappa\lambda}(X\,;\,Y)$ has the same form as above with respect to a's according as $n = 2$, $n = 3$ or $n \geqq 4$. Now put

$$A = \begin{pmatrix} a^1_1 & a^1_2 & 0 & \ldots & 0 & 0 \\ a^2_1 & a^2_2 & 0 & \ldots & 0 & 0 \\ 0 & a^3_2 & a^3_3 & \ldots & 0 & 0 \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a^{n-1}_{n-1} & 0 \\ 0 & 0 & 0 & \ldots & a^n_{n-1} & a^n_n \end{pmatrix}$$

and
$$x^1 = x^3 = \ldots\ldots = y^2 = y^4 = \ldots\ldots = 1\,,$$
$$x^2 = x^4 = \ldots\ldots = y^1 = y^3 = \ldots\ldots = 0\,.$$

Then if $n =$ even, we have from above

$$w^{\varkappa\lambda}(a^1_1, a^2_1, a^3_3, a^4_3, \ldots\ldots, a^n_{n-1}, a^1_2, a^2_2, a^3_2, a^4_4, \ldots\ldots, a^n_n)$$

$=$ a H. E. of third degree, etc. according to the value of n. Hence we can enunciate as follows

$w^{\varkappa\lambda}(x\,;y)$ must necessarily be a H. E. of third degree of x^1, x^2; y^1, y^2 and so on according to the value of n.

If $n=$ odd, a similar treatment shows that the same result may be obtained.

Whether this condition is sufficient or not will be studied in what follows.

(A.1) Case $n=2$, and

$$w^{\varkappa\lambda} = \text{a H. E. of third degree,}$$

$$= \sum A^{\varkappa\lambda}_{ijk}x^i x^j x^k + \sum B^{\varkappa\lambda}_{ijk}x^i x^j y^k + \sum C^{\varkappa\lambda}_{ijk}x^i y^j y^k + \sum D^{\varkappa\lambda}_{ijk}y^i y^j y^k\,,$$

where A's, D's are symmetric with respect to i, j, k and B's, C's are symmetric with respect to i, j and j, k respectly. (2.1) becomes

$$\sum A^{\varkappa\lambda}_{rst}(\sum a^r_i x^i)(\sum a^s_j x^j)(\sum a^t_k x^k) + \sum B^{\varkappa\lambda}_{rst}(\sum a^r_i x^i)(\sum a^s_j x^j)(\sum a^t_k y^k) + \ldots\ldots$$

$$= \sum K^{\varkappa\lambda}_{\mu\nu}(\sum A^{\mu\nu}_{ijk}x^i x^j x^k + \sum B^{\mu\nu}_{ijk}x^i x^j y^k + \ldots\ldots)\,,$$

that is

(a) $$\sum A^{\varkappa\lambda}_{rst}a^{\lambda}_i a^s_j a^t_k = \sum K^{\varkappa\lambda}_{\mu\nu}A^{\mu\nu}_{ijk}\,,$$

(b) $$\sum B^{\varkappa\lambda}_{rst}a^r_i a^s_j a^t_k = \sum K^{\varkappa\lambda}_{\mu\nu}B^{\mu\nu}_{ijk}\,,$$

(c) $$\sum C^{\varkappa\lambda}_{rst}a^r_i a^s_j a^t_k = \sum K^{\varkappa\lambda}_{\mu\nu}C^{\mu\nu}_{ijk}\,,$$

(d) $$\sum D^{\varkappa\lambda}_{rst}a^r_i a^s_j a^t_k = \sum K^{\varkappa\lambda}_{\mu\nu}D^{\mu\nu}_{ijk}\,.$$

In order that $\sum A^{\varkappa\lambda}_{ijk}x^i x^j x^k$ does not vanish, the matrix $(A^{\varkappa\lambda}_{ijk})$ must be regular. For, if $|A| = 0$ the four expressions in the left-hand side of (a) must be linearly dependent. Thus we have $A^{\varkappa\lambda}_{ijk} = 0$.

Now since the matrix $(A^{\varkappa\lambda}_{ijk})$ is assumed to be regular, K's are determined from (a). If these K's thus determined satisfy the other equality in (b), (c) or (d), we have for example

$$\lambda \sum B^{\varkappa\lambda}_{rst}a^r_1 a^s_1 a^t_2 = \mu \sum A^{\varkappa\lambda}_{rst}a^r_1 a^s_1 a^t_2$$

$$\therefore \quad \sum (\lambda B^{\varkappa\lambda}_{rst} - \mu A^{\varkappa\lambda}_{rst})a^r_1 a^s_1 a^t_2 = 0$$

$$\therefore \quad \lambda B_{rst} = \mu A_{rst}\,.$$

Thus B's must also be symmetric and proportional to A's. Generally we have

$$\frac{A^{\kappa\lambda}_{ijk}}{a}=\frac{B^{\kappa\lambda}_{ijk}}{b}=\frac{C^{\kappa\lambda}_{ijk}}{c}=\frac{D^{\kappa\lambda}_{ijk}}{d}=a^{\kappa\lambda}_{ijk}$$

$$\therefore\quad \boxed{\begin{aligned} w^{\kappa\lambda}&=\sum a^{\kappa\lambda}_{ijk}(ax^ix^jx^k+bx^ix^jy^k+cx^iy^jy^k+dy^iy^jy^k)\\ &=a\sum a^{\kappa\lambda}_{ijk}(x^i-ly^i)(x^j-my^j)(x^k-ny^k)\end{aligned}}$$

where l, m, n are the three roots of the cubic equation

$$az^3+bz^2+cz+d=0\,.$$

If $\sum A^{\kappa\lambda}_{ijk}x^ix^jx^k$, $\sum D^{\kappa\lambda}_{ijk}y^iy^jy^k$ do not appear and B's, C's are not symmetric,[1] we have:—

In relation to (b), if

$$\text{(2.3)}\qquad \begin{vmatrix} B^{11}_{111} & B^{12}_{111} & B^{21}_{111} & B^{22}_{111}\\ B^{11}_{112} & B^{12}_{112} & B^{21}_{112} & B^{22}_{112}\\ B^{11}_{221} & B^{12}_{221} & B^{21}_{221} & B^{22}_{221}\\ B^{11}_{222} & B^{12}_{222} & B^{21}_{222} & B^{22}_{222}\end{vmatrix}=0$$

$$\sum B^{\kappa\lambda}_{rst}a_1a_1a_1\,,\qquad \sum B^{\kappa\lambda}_{rst}a^r_1a^s_1a^t_2\,,\qquad \sum B^{\kappa\lambda}_{rst}a^r_2a^s_2a^t_1\,,\qquad \sum B^{\kappa\lambda}_{rst}a^r_2a^s_2a^t_2$$

are linearly dependent and hence one of them must vanish; thus we have

$$B^{\kappa\lambda}_{111}=B^{\kappa\lambda}_{222}=0\,,\qquad B^{\kappa\lambda}_{112}+2B^{\kappa\lambda}_{121}=0\,,\qquad B^{\kappa\lambda}_{221}+2B^{\kappa\lambda}_{122}=0\ ^{(2)}$$

or $B^{\kappa\lambda}_{ijk}=0$. In the former case (b) reduces to two equations; the latter case, being trivial, is excluded. If the determinant (2.3) is not zero, K's are determined from the corresponding four equations in (b). In order that the remaining two may be censistent with the four, it is necessary and sufficient that $\sum B^{\kappa\lambda}_{rst}a^r_1a^s_2a^t_1$ and $\sum B^{\kappa\lambda}_{rst}a^r_1a^s_2a^t_2$ be expressed as a linear combination of the corresponding expressions in (b). Hence we have

(1) If either B's, C's are symmetric, w must be of the same form as above.

(2) In this case we say that B's are cyclically zero (with respect to their lower suffices).

$$(2.4)\quad \begin{aligned} &\varkappa \sum B^{\varkappa\lambda}_{rst} a^r_1 a^s_2 a^t_1 = \lambda \sum B^{\varkappa\lambda}_{rst} a^r_1 a^s_1 a^t_2 , \\ \text{and}\quad &\mu \sum B^{\varkappa\lambda}_{rst} a^r_2 a^s_1 a^t_2 = \nu \sum B^{\varkappa\lambda}_{rst} a^r_2 a^s_2 a^t_1 , \end{aligned}$$

where $\varkappa, \lambda, \mu, \nu$ are constant. From (2.4) we get

$$\varkappa B_{111} = \lambda B_{111}, \qquad \varkappa B_{112} = (2\lambda - \varkappa) B_{121}, \qquad \varkappa B_{121} = \lambda B_{112};$$

$$\varkappa B_{222} = \lambda B_{222}, \qquad \varkappa B_{221} = (2\lambda - \varkappa) B_{212}, \qquad \varkappa B_{212} = \lambda B_{221}.$$

$$\therefore \quad \begin{vmatrix} \varkappa & 2\lambda - \varkappa \\ \lambda & \varkappa \end{vmatrix} = 0, \qquad \text{that is,} \qquad \varkappa = \lambda \qquad \text{or} \qquad \varkappa = -2\lambda$$

or $B^{\varkappa\lambda}_{rst} = 0$. If $\varkappa = \lambda$, B's must be symmetric, and if $\varkappa = -2\lambda$, B's must be cyclically zero. At any rate, the determinant (2.3) vanishes, contradicting our assumption. Thus the only case where B's are cyclically zero is possible. Similarly C's are cyclically zero. The four equations, two from (b) and two from (c), determine the K's. And in this case we obtain an expression which contradicts with $V(x) \neq 0$:

$$w^{\varkappa\lambda} = (a^{\varkappa\lambda}_1 x^1 + a^{\varkappa\lambda}_2 x^2 + b^{\varkappa\lambda}_1 y^1 + b^{\varkappa\lambda}_2 y^2)(x^1 y^2 - x^2 y^1).$$

(A.2) Case $n = 2$, and

$$\begin{aligned} w^{\varkappa\lambda} &= \text{a H. E. of second degree} + \text{an absolute constant}, \\ &= A^{\varkappa\lambda} + \sum A^{\varkappa\lambda}_{ij} x^i x^j + \sum B^{\varkappa\lambda}_{ij} x^i y^j + \sum C^{\varkappa\lambda}_{ij} y^i y^j , \end{aligned}$$

where A's and C's are symmetric with respect to their lower suffices, We obtain as before

$$(0) \qquad A^{\varkappa\lambda} = \sum K^{\varkappa\lambda}_{\mu\nu} A^{\mu\nu},$$

$$(a) \qquad \sum A^{\varkappa\lambda}_{s} a^r_i a^s_j = \sum K^{\varkappa\lambda}_{\mu\nu} A^{\mu\nu}_{ij},$$

$$(b) \qquad \sum B^{\varkappa\lambda}_{rs} a^r_i a^s_j = \sum K^{\varkappa\lambda}_{\mu\nu} B^{\mu\nu}_{ij},$$

$$(c) \qquad \sum C^{\varkappa\lambda}_{rs} a^r_1 a^s_j = \sum K^{\varkappa\lambda}_{\mu\nu} C^{\mu\nu}_{ij}.$$

First we assume that $\sum B^{\varkappa\lambda}_{ij} x^i y^j$ appears. If $|B| \neq 0$, K's are determined from (b).[1] Hence we have

(1) Since K's are quadratic forms of a's, (0) cannot be satisfied, that is to say, an absolute term cannot exist in this case.

$$\boxed{w^{\varkappa\lambda} = \text{any bilinear form.}}$$

In order that the other term may appear, K's thus determined should satisfy the other equations, say (a). Then have

$$\lambda \sum A^{\varkappa\lambda}_{rs} a^r_1 a^s_1 = \mu \sum B^{\varkappa\lambda}_{rs} a^r_1 a^s_1$$

$$\therefore \qquad \lambda A^{\varkappa\lambda}_{rs} = \mu \frac{B^{\varkappa\lambda}_{rs} + B^{\varkappa\lambda}_{sr}}{2},$$

and

$$\varkappa \sum A^{\varkappa\lambda}_{rs} a^r_1 a^s_2 = \nu \sum B^{\varkappa\lambda}_{rs} a^r_1 a^s_2 + \varepsilon \sum B^{\varkappa\lambda}_{rs} a^r_2 a^s_1 ,$$

that is

$$\varkappa A_{rs} = \nu B_{rs} + \varepsilon B_{sr} .$$

$$\therefore \qquad \frac{B_{rs}}{B_{sr}} = \text{const.} = \pm 1 .$$

Thus B's must be symmetric or alternating. But this is contradictory to the non-vanishing of $|B|$. If $|B| = 0$, we have

$$\sum B^{\varkappa\lambda}_{rs} a^r_1 a^s_1 = 0 \qquad \text{or} \qquad \sum B^{\varkappa\lambda}_{rs} a^r_2 a^s_2 = 0 ,$$

or

$$\lambda \sum B^{\varkappa\lambda}_{rs} a^r_1 a^s_2 = \mu \sum B^{\varkappa\lambda}_{rs} a^r_2 a^s_1 .$$

From the former two relations, we see that B's are alternating and (b) reduces to a single equation. From the last relation, we get $\lambda B_{rs} = \mu B_{sr}$, that is to say, B's must be either symmetric or alternating.

On the other hand, the consistency of (a) and (c) requires that

$$(A, C) = (A_{11}, A_{22} ; C_{11}, C_{22}) = \begin{vmatrix} A^{11}_{11} & A^{12}_{11} & A^{21}_{11} & A^{22}_{11} \\ A^{11}_{22} & A^{12}_{22} & A^{21}_{22} & A^{22}_{22} \\ C^{11}_{11} & C^{12}_{11} & C^{21}_{11} & C^{22}_{11} \\ C^{11}_{22} & C^{12}_{22} & C^{21}_{22} & C^{22}_{22} \end{vmatrix} = 0 ,$$

since if $(A, C) \neq 0$, we have $\sum A^{\varkappa\lambda}_{rs} a^r_1 a^s_2 = 0$ that is $A^{\varkappa\lambda}_{rs} = 0$, which cannot be allowed. From $(A, C) \neq 0$, we have

$$\varkappa \sum A^{\varkappa\lambda}_{rs} a^r_1 a^s_1 = \lambda \sum C^{\varkappa\lambda}_{rs} a^r_1 a^s_1$$

$$\therefore \qquad \varkappa A^{\varkappa\lambda}_{rs} = \lambda C^{\varkappa\lambda}_{rs} .$$

In this case (a) $\big($or (c)$\big)$ is only to be taken as independent.

Now returning to our present case, if B's are alternating, and $\sum A_{ij}^{\kappa\lambda}x^ix^j+\sum C_{ij}^{\kappa\lambda}y^iy^j$ exist and $(A_{11}, A_{12}, A_{22}, B_{12})=0$, we have $\lambda\sum B_{rs}^{\kappa\lambda}a_1^ra_2^s=\mu\sum A_{rs}^{\kappa\lambda}a_1^ra_2^s$, that is $\mu A_{rs}=\dfrac{(B_{rs}+B_{sr})}{2}=0$, which is a contradiction. Hence $(A_{11}, A_{12}, A_{22}, B_{12})\neq 0$. $A^{\kappa\lambda}$ cannot appear in this case.

$$\boxed{w^{\kappa\lambda}=\sum a_{ij}^{\kappa\lambda}(ax^ix^j+cy^iy^j)+B^{\kappa\lambda}(x^1y^2-x^2y^1)\,.}$$

If $\sum A_{ij}^{\kappa\lambda}x^ix^j+\sum C_{ij}^{\kappa\lambda}y^iy^j$ do not exist

$$\boxed{w^{\kappa\lambda}=A^{\kappa\lambda}+B^{\kappa\lambda}(x^1y^2-x^2y^1)\,.}$$

When B's are symmetric, from the above considerations we have

$$\boxed{\begin{aligned} w^{\kappa\lambda}&=A^{\kappa\lambda}+\sum a_{ij}^{\kappa\lambda}(ax^ix^j+bx^iy^j+cy^iy^j)\\ &=A^{\kappa\lambda}+a\sum a_{ij}^{\kappa\lambda}(x^i-ly^i)(x^j-my^j)\,.\end{aligned}}$$

Secondly if $\sum B_{ij}^{\kappa\lambda}x^iy^j$ does not appear, we have

$$w^{\kappa\lambda}=A^{\kappa\lambda}+\sum a_{ij}^{\kappa\lambda}(ax^ix^j+cy^iy^j)\,,$$

which may be included in the foregoing result.

(A.3) Case $n=2$, and

$$w^{\kappa\lambda}=A^{\kappa\lambda}+\sum A_i^{\kappa\lambda}x^i+\sum B_i^{\kappa\lambda}y^i\,.$$

We get as before

$$\text{(0)}\qquad A^{\kappa\lambda}=\sum K_{\mu\nu}^{\kappa\lambda}A^{\mu\nu}$$

$$\text{(a)}\qquad \sum A_r^{\kappa\lambda}a_i^r=\sum K_{\mu\nu}^{\kappa\lambda}A_i^{\mu\nu}$$

$$\text{(b)}\qquad \sum B_r^{\kappa\lambda}a_i^r=\sum K_{\mu\nu}^{\kappa\lambda}B_i^{\mu\nu}\,.$$

If $(A_1, A_2, B_1, B_2)\neq 0$, an absolute term cannot appear.

$$\boxed{w^{\kappa\lambda}=\sum A_i^{\kappa\lambda}x^i+\sum B_i^{\kappa\lambda}y^i\,.}$$

If $(A_1, A_2, B_1, B_2)=0$, $\lambda\sum A_r^{\kappa\lambda}a_1^r=\mu\sum B_r^{\kappa\lambda}a_1^r$ that is $\lambda A_r^{\kappa\lambda}=\mu B_r^{\kappa\lambda}$.

(B.1) $n = 3$, $w^{\kappa\lambda} = A^{\kappa\lambda} + \sum A_{ij}^{\kappa\lambda} x^i x^j + \sum B_{ij}^{\kappa\lambda} x^i y^j + \sum C_{ij}^{\kappa\lambda} y^i y^j$.

We have

$$\text{(0)} \qquad A^{\kappa\lambda} = \sum K_{\mu\nu}^{\kappa\lambda} A^{\mu\nu},$$

$$\text{(a)} \qquad \sum A_{rs}^{\kappa\lambda} a_i^r a_j^s = \sum K_{\mu\nu}^{\kappa\lambda} A_{ij}^{\mu\nu},$$

$$\text{(b)} \qquad \sum B_{rs}^{\kappa\lambda} a_i^r a_j^s = \sum K_{\mu\nu}^{\kappa\lambda} B_{ij}^{\mu\nu},$$

$$\text{(c)} \qquad \sum C_{rs}^{\kappa\lambda} a_i^r a_j^s = \sum K_{\mu\nu}^{\kappa\lambda} C_{ij}^{\mu\nu}.$$

If $|B| \neq 0$, an absolute term cannot appear. In order that $\sum A_{ij}^{\kappa\lambda} x^i x^j + \sum C_{ij}^{\kappa\lambda} y^i y^j$ may appear, we must have for example

$$\lambda \sum A_{rs}^{\kappa\lambda} a_1^r a_1^s = \mu \sum B_{rs}^{\kappa\lambda} a_1^r a_1^s,$$

or

$$\lambda \sum A_{rs}^{\kappa\lambda} a_1^r a_2^s = \mu \sum B_{rs}^{\kappa\lambda} a_1^r a_2^s + \nu \sum B_{rs}^{\kappa\lambda} a_2^r a_1^s,$$

that is,

$$\lambda A_{rs} = \mu \frac{B_{rs} + B_{sr}}{2}$$

or

$$\lambda A_{rs} = \mu B_{rs} + \nu B_{sr} = \mu B_{sr} + \nu B_{rs} = \mu (B_{rs} + B_{sr}).$$

At any rate

$$A_{rs} = a \cdot \frac{B_{rs} + B_{sr}}{2}.$$

Similarly

$$C_{rs} = c \cdot \frac{B_{rs} + B_{sr}}{2}.$$

$$\boxed{\begin{aligned} w &= \sum B_{ij}^{\kappa\lambda}(a x^i x^j + x^i y^j + c y^i y^j) \\ &= \sum B_{ij}^{\kappa\lambda}(x^i - l y^i)(x^j - m y^j)^{(1)} \end{aligned}}$$

If $|B| = 0$, we have $\sum B_{rs}^{\kappa\lambda} a_i^r a_i^s = 0$ or $\lambda \sum B_{rs}^{\kappa\lambda} a_i^r a_j^s = \mu \sum B_{rs}^{\kappa\lambda} a_j^r a_i^s$. From the first we see that B's are alternating, and from the second we see that B's are either alternating or symmetric. Hence $|B| = 0$ occurs when and only when B's are symmetric or alternating.

If B's are symmetric, (b) reduces to six equations and the similar observations of the case $|B| \neq 0$ show that

$$A_{rs} = a \frac{B_{rs} + B_{sr}}{2} = a B_{rs}.$$

(1) We shall call such a form a general bilinear form.

Similarly $$C_{rs} = cB_{rs}.$$

$$w = A^{\varkappa\lambda} + \sum B_{ij}^{\varkappa\lambda}(ax^i x^j + x^i y^j + cy^i y^j)$$
$$= \text{const.} + \text{a general symmetric bilinear form.}$$

If B'e are alternating, (b) reduces to three equations. If further $(A_{11}, A_{12}, A_{22}, A_{13}, A_{23}, A_{33}, B_{12}, B_{13}, B_{23}) \neq 0$,

$$w^{\varkappa\lambda} = \sum a_{ij}^{\varkappa\lambda}(ax^i x^j + cy^i y^j) + \begin{vmatrix} a_1^{\varkappa\lambda} & a_2^{\varkappa\lambda} & a_3^{\varkappa\lambda} \\ x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{vmatrix},$$

or

$$w^{\varkappa\lambda} = A^{\varkappa\lambda} + \begin{vmatrix} a_1^{\varkappa\lambda} & a_2^{\varkappa\lambda} & a_3^{\varkappa\lambda} \\ x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{vmatrix},$$

according as the term $\sum A_{ij}^{\varkappa\lambda} x^i x^j + \sum B_{ij}^{\varkappa\lambda} y^i y^j$ appears or not.

If $(A_{11}, A_{12}, A_{22}, A_{13}, A_{23}, A_{33}, B_{12}, B_{13}, B_{23}) = 0$, we have

$$\lambda \sum A_{rs}^{\varkappa\lambda} a_1^r a_2^s = \mu \sum B_{rs}^{\varkappa\lambda} a_1^r a_2^s, \qquad \text{that is,} \qquad \lambda A_{rs} = \mu B_{rs}.$$

This means that B's are also symmetric, which is not allowable.

If terms $\sum B_{ij}^{rs} x^i y^j$ do not exist, we easily get

$$w^{\varkappa\lambda} = A^{\varkappa\lambda} + \sum B_{ij}^{\varkappa\lambda}(ax^i x^j + cy^i y^j).$$

(B.2) $\quad n = 3, \quad w^{\varkappa\lambda} = A^{\varkappa\lambda} + \sum A_i^{\varkappa\lambda} x^i + \sum B_i^{\varkappa\lambda} y^i.$

We have

(0) $$A^{\varkappa\lambda} = \sum K_{\mu\nu}^{\varkappa\lambda} A^{\mu\nu},$$

(a) $$\sum A_r^{\varkappa\lambda} a_i^r = \sum K_{\mu\nu}^{\varkappa\lambda} A_i^{\mu\nu},$$

(b) $$\sum B_r^{\varkappa\lambda} a_i^r = \sum K_{\mu\nu}^{\varkappa\lambda} B_i^{\mu\nu}.$$

K's are always determined.

$$\boxed{w^{\varkappa\lambda} = \text{const.} + \text{linear form.}}$$

(B.3) $n = 3$,

$$w^{\varkappa\lambda} = \sum A_i^{\varkappa\lambda} x^i + \sum B_i^{\varkappa\lambda} y^i + \sum A_{ij}^{\varkappa\lambda} x^i x^j + \sum B_{ij}^{\varkappa\lambda} x^i y^j + \sum C_{ij}^{\varkappa\lambda} y^i y^j .$$

We have

(a′) $\sum A_r^{\varkappa\lambda} a_i^r = \sum K_{\mu\nu}^{\varkappa\lambda} A_i^{\mu\nu}$,

(b′) $\sum B_r^{\varkappa\lambda} a_i^r = \sum K_{\mu\nu}^{\varkappa\lambda} B_i^{\mu\nu}$,

(a) $\sum A_{rs}^{\varkappa\lambda} a_i^r a_j^s = \sum K_{\mu\nu}^{\varkappa\lambda} A_{ij}^{\mu\nu}$,

(b) $\sum B_{rs}^{\varkappa\lambda} a_i^r a_j^s = \sum K_{\mu\nu}^{\varkappa\lambda} B_{ij}^{\mu\nu}$,

(c) $\sum C_{rs}^{\varkappa\lambda} a_i^r a_j^s = \sum K_{\mu\nu}^{\varkappa\lambda} C_{ij}^{\mu\nu}$.

If $\sum B_{ij} x^i x^j$ appear, and $|B| \neq 0$, linear terms cannot exist and the similar observations of the foregoing case show that

$$\boxed{w^{\varkappa\lambda} = \text{general bilinear form.}}$$

If, on the contrary, $|B| = 0$, B's must be symmetric or alternating. When B's are symmetric, A's, B's and C's should be proportional. In order that the linear terms should appear A_i's must be proportional to B_i's.

$$\boxed{w^{\varkappa\lambda} = \sum a_i^{\varkappa\lambda}(ax^i + by^i) + \sum a_{ij}^{\varkappa\lambda}(x^i - ly^i)(x^i - my^i) .}$$

When B's are alternating

$$\boxed{w^{\varkappa\lambda} = \sum A_i^{\varkappa\lambda} x^i + \sum B_i^{\varkappa\lambda} y^i + \begin{vmatrix} a_1^{\varkappa q} & a_2^{\varkappa b} & a_3^{\varkappa b} \\ x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{vmatrix} ,}$$

if $\sum A_{ij}^{\varkappa\lambda} x^i x^j + \sum C_{ij}^{\varkappa\lambda} y^i y^j$ do not appear. And if the case is contrary, since $(A_{11}, A_{12}, A_{22}, A_{13}, A_{23}, A_{33}, B_{12}, B_{13}, B_{23})$ cannot vanish, linear terms do not occur.

$$w^{\kappa\lambda} = \sum a_{ij}^{\kappa\lambda}(ax^ix^j + cy^iy^j) + \begin{vmatrix} a_1^{\kappa\lambda} & a_2^{\kappa\lambda} & a_3^{\kappa\lambda} \\ x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \end{vmatrix}.$$

If $\sum B_{ij}^{\kappa\lambda}x^iy^j$ do not exist, we have

$$w^{\kappa\lambda} = \sum a_i^{\kappa\lambda}(\bar{a}x^i + \bar{b}y^i) + \sum a_{ij}^{\kappa\lambda}(ax^ix^j + cy^iy^j)$$

which is included in the fore-going result.

(C) $n \geqq 4$,

$$w^{\kappa\lambda} = A^{\kappa\lambda} + \sum A_i^{\kappa\lambda}x^i + \sum B_i^{\kappa\lambda}y^i + \sum A_{ij}^{\kappa\lambda}x^ix^j + \sum B_{ij}^{\kappa\lambda}x^iy^j + \sum C_{ij}^{\kappa\lambda}y^iy^j.$$

We have

(0) $$A^{\kappa\lambda} = \sum K_{\mu\nu}^{\kappa\lambda}A^{\mu\nu},$$

(a) $$\sum A_r^{\kappa\lambda}a_i^r = \sum K_{\mu\nu}^{\kappa\lambda}A_i^{\mu\nu},$$

(b) $$\sum B_r^{\kappa\lambda}a_i^r = \sum K_{\mu\nu}^{\kappa\lambda}B_i^{\mu\nu},$$

(a) $$\sum A_{rs}^{\kappa\lambda}a_i^ra_j^s = \sum K_{\mu\nu}^{\kappa\lambda}A_{ij}^{\mu\nu},$$

(b) $$\sum B_{rs}^{\kappa\lambda}a_i^ra_j^s = \sum K_{\mu\nu}^{\kappa\lambda}B_{ij}^{\mu\nu},$$

(c) $$\sum C_{rs}^{\kappa\lambda}a_i^ra_j^s = \sum K_{\mu\nu}^{\kappa\lambda}C_{ij}^{\mu\nu}.$$

When $\sum B_{ij}^{\kappa\lambda}x^iy^j$ occur and $|B| \neq 0$, then neither the absolute nor the linear term can appear.

$$w^{\kappa\lambda} = \text{general bilinear form.}$$

If $|B| = 0$, B's must be symmetric or alternating. If B's are symmetric, A's, B's and C's must be proportional. Since

$$1 + 2n + \frac{n(n+1)}{2} \gtrless n^2$$

according as $n = 4, 5$ or $n \geqq 6$, corresponding to $n \geqq 6$ we have

$$w^{\kappa\lambda} = A^{\kappa\lambda} + \sum A_i^{\kappa\lambda}x^i + \sum B_i^{\kappa\lambda}y^i + \sum a_{ij}^{\kappa\lambda}(x^i - ly^i)(x^j - my^j).$$

If $n = 4$ or 5

$$w^{\varkappa\lambda} = A^{\varkappa\lambda} + \sum a_i^{\varkappa\lambda}(ax^i + by^i) + \sum a_{ij}^{\varkappa\lambda}(x^i - ly^i)(x^j - my^j).$$

Moreover, especially, for $n = 5$ it may occur that

$$w^{\varkappa\lambda} = \sum A_i^{\varkappa\lambda} x^i + \sum B_i^{\varkappa\lambda} y^i + \sum a_{ij}^{\varkappa\lambda}(x^i - ly^i)(x^j - my^j).$$

If B's are alternating, (b) reduces to $\frac{n(n-1)}{2}$ equations. Now in the case where $\sum A_{ij}^{\varkappa\lambda} x^i x^j + \sum C_{ij}^{\varkappa\lambda} y^i y^j$ does not occur, since $\frac{n(n-1)}{2} + 2n + 1 < n^2$, we have

$$w^{\varkappa\lambda} = A^{\varkappa\lambda} + \sum A_i^{\varkappa\lambda} x^i + \sum B_i^{\varkappa\lambda} y^i + \sum B_{ij}^{\varkappa\lambda} \begin{vmatrix} x^i & x^j \\ y^i & y^j \end{vmatrix}.$$

On the contrary if $\sum A_{ij}^{\varkappa\lambda} x^i x^j + \sum C_{ij}^{\varkappa\lambda} y^i y^j$ appears in $w^{\varkappa\lambda}$ $(A_{11}, A_{12}, A_{22}, \ldots, B_{12}, B_{13}, B_{23}, \ldots)$ cannot vanish in so for as $\sum B_{ij}^{\varkappa\lambda} x^i y^j$ exists. And if $(A_{11}, A_{12}, A_{22}, A_{13}, \ldots, B_{12}, B_{13}, \ldots) \neq 0$, the absolute term and linear terms cannot occur in $w^{\varkappa\lambda}$. Hence we have

$$w^{\varkappa\lambda} = \sum a_{ij}^{\varkappa\lambda}(ax^i x^j + cy^i y^j) + \sum B_{ij}^{\varkappa\lambda} \begin{vmatrix} x^i & x^j \\ y^i & y^j \end{vmatrix}.$$

Finally, if $\sum B_{ij}^{\varkappa\lambda} x^i y^j$ does not appear in w, the result can take no other form than what has already been obtained.